Constraint Programming in a Nutshell

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Optimisation is a science of **service**: to scientists, to engineers, to artists, and to society.
Outline

1. Constraint Problems
2. Constraint Programming Technology
3. CP Modelling
4. CP Solving
   - Systematic Search
   - Local Search
5. History of CP
6. Success Stories of CP
7. When (Not) to Use CP?
8. Bibliography
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### Example (Agricultural experiment design)

<table>
<thead>
<tr>
<th></th>
<th>plot1</th>
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<th>plot6</th>
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**Constraints to be satisfied:**

1. Equal sample size: Every grain is grown in 3 plots.
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<tr>
<th>Mon</th>
<th>Tue</th>
<th>Wed</th>
<th>Thu</th>
<th>Fri</th>
<th>Sat</th>
<th>Sun</th>
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</thead>
<tbody>
<tr>
<td>Doctor A</td>
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<tr>
<td>Doctor B</td>
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<td>Doctor C</td>
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<td>Doctor D</td>
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<tr>
<td>Doctor E</td>
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</tbody>
</table>

**Constraints to be satisfied:**

1. \#doctors-on-call / day = 1
2. \#operations / workday \( \leq \) 2
3. \#operations / week \( \geq \) 7
4. \#appointments / week \( \geq \) 4
5. day off after operation day
6. . . .

**Objective function to be minimised:**

- Cost: . . .
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<th></th>
<th>Mon</th>
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<th>Fri</th>
<th>Sat</th>
<th>Sun</th>
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</thead>
<tbody>
<tr>
<td>Doctor A</td>
<td>call</td>
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<td>oper</td>
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<td>oper</td>
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<tr>
<td>Doctor B</td>
<td>app</td>
<td>call</td>
<td>—</td>
<td>oper</td>
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<td>call</td>
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<td>Doctor C</td>
<td>oper</td>
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<td>app</td>
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<td>—</td>
<td>oper</td>
<td>—</td>
<td>call</td>
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<td>—</td>
</tr>
</tbody>
</table>

**Constraints to be satisfied:**

1. #doctors-on-call / day = 1
2. #operations / workday ≤ 2
3. #operations / week ≥ 7
4. #appointments / week ≥ 4
5. day off after operation day
6. ...

**Objective function to be minimised:**

- Cost: ...
Example (Vehicle routing: parcel delivery)

**Given** a depot with a vehicle fleet and parcels for clients, **find** which vehicle brings which parcel to which client when.

**Constraints** to be **satisfied**:
1. All parcels are delivered on time.
2. No vehicle is overloaded.
3. Driver regulations are respected.
4. . .

**Objective function to be minimised**:
- Cost: the total fuel consumption and driver salary.

Example (Travelling salesperson: optimisation TSP)

**Given** a map and cities, **find** a **shortest** route visiting each city once and returning to the starting city.
Applications in Air Traffic Management

Demand vs capacity

Airspace sectorisation

Contingency planning

<table>
<thead>
<tr>
<th>Flow</th>
<th>Time Span</th>
<th>Hourly Rate</th>
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</thead>
<tbody>
<tr>
<td>From: Arlanda</td>
<td>00:00 – 09:00</td>
<td>3</td>
</tr>
<tr>
<td>To: west, south</td>
<td>09:00 – 18:00</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>18:00 – 24:00</td>
<td>2</td>
</tr>
<tr>
<td>From: Arlanda</td>
<td>00:00 – 12:00</td>
<td>4</td>
</tr>
<tr>
<td>To: east, north</td>
<td>12:00 – 24:00</td>
<td>3</td>
</tr>
</tbody>
</table>

Workload balancing
Example (Airspace sectorisation)

**Given** an airspace split into $c$ cells, and a targeted number $s$ of sectors.

**Find** a colouring of the cells into $s$ connected convex sectors, with minimal imbalance of the workloads of their air traffic controllers.

There are $s^c$ possible colourings, but very few optimally satisfy the constraints: is intelligent search necessary?
Applications in Biology and Medicine

Phylogenetic supertree

- Pseudorealis adleri
- Eucalyptus minor
- Magelathodes antipodes
- Eucalyptus pauciflora
- Phyllocladus manescui
- D. crassifolia
- Eucalyptus macrocarpa
- Helianthus annuus
- Phyllocladus glaucus
- Eucalyptus nitens
- Phyllocladus banksii
- Eucalyptus pauciflora
- Phyllocladus glaucus
- Eucalyptus nitens
- Phyllocladus banksii
- Eucalyptus pauciflora
- Phyllocladus manescui
- D. crassifolia
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Haplotype inference

- Oxytropis alpina
- Phalaenopsis amabilis
- Eucalyptus nitens
- Phyllocladus banksii
- Eucalyptus pauciflora
- Phyllocladus manescui
- D. crassifolia
- Eucalyptus macrocarpa
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Medical image analysis

Doctor rostering
Applications in Programming and Testing

Robot-task sequencing

Sensor-net configuration

Compiler design

Base-station testing
Other Application Areas

School timetabling

<table>
<thead>
<tr>
<th>Monday</th>
<th>Tuesday</th>
<th>Wednesday</th>
<th>Thursday</th>
<th>Friday</th>
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<tr>
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<td>LAB220</td>
<td>VT208</td>
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<td>Computer Graphics</td>
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Sports tournament design

Security: SQL injection?

Container packing
Definition

In a constraint problem, values have to be found for all the unknowns, called variables (in the mathematical sense) and ranging over given sets called domains, so that:

- All the given constraints on the variables are satisfied.
- Optionally: A given objective function on the variables has an optimal value: minimal cost or maximal profit.

Definition

A candidate solution to a constraint problem assigns to each variable a value within its domain. The search space consists of all candidate solutions.
Example (Optimisation TSP over \( n \) cities)

A brute-force algorithm evaluates all \( n! \) candidate routes:

- A computer of today evaluates \( 10^6 \) routes / second:

<table>
<thead>
<tr>
<th>( n )</th>
<th>Time</th>
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<tbody>
<tr>
<td>11</td>
<td>40 seconds</td>
</tr>
<tr>
<td>14</td>
<td>1 day</td>
</tr>
<tr>
<td>18</td>
<td>203 years</td>
</tr>
<tr>
<td>20</td>
<td>77k years</td>
</tr>
</tbody>
</table>

- Planck time is shortest useful interval: \( \approx 5.4 \times 10^{-44} \) s; a Planck computer would evaluate \( 1.8 \times 10^{43} \) routes / s:

<table>
<thead>
<tr>
<th>( n )</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>37</td>
<td>0.7 seconds</td>
</tr>
<tr>
<td>41</td>
<td>20 days</td>
</tr>
<tr>
<td>48</td>
<td>1.5 \times \text{age of universe}</td>
</tr>
</tbody>
</table>

The dynamic program by Bellman-Held-Karp “only” takes \( O(n^2 \cdot 2^n) \) time: a computer of today takes a day for \( n = 27 \), a year for \( n = 35 \), the age of the universe for \( n = 67 \), and it beats the \( O(n!) \) algo on the Planck computer for \( n \geq 44 \).
Search spaces are often larger than the universe!

Many important real-life problems are NP-hard and can only be solved exactly & fast enough by *intelligent* search, unless $P = NP$:

NP-hardness is not where the fun ends, but where it begins!
Do not give up but try to stay ahead of the curve: there is an instance size until which an exact algorithm is fast enough!

The Concorde TSP Solver beats the Bellman-Held-Karp exact algo: it uses approximation & local-search algorithms, but it can sometimes prove the exactness (optimality) of its solutions. The largest instance it has solved exactly, in 136 CPU years in 2006, has 85,900 cities! Let the fun begin!
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A solving technology offers methods and tools for:

what: Modelling constraint problems in declarative language.

and / or

how: Solving constraint problems intelligently:

- **Search**: Explore the space of candidate solutions.
- **Inference**: Reduce the space of candidate solutions.
- **Relaxation**: Exploit solutions to easier problems.

A solver is a software that takes a model as input and tries to solve the modelled problem.

Combinatorial (= discrete) optimisation covers satisfaction and optimisation problems, for variables over discrete sets.

The ideas in this lecture extend to continuous optimisation, to soft optimisation, and to stochastic optimisation.
Modelling vs Programming

problem

specification

what? (declarative)

model

automatic!

algorithm

how? (imperative)

program

manual!
Example (Solving technologies)

With general-purpose solvers, taking a model as input:
- Boolean satisfiability (SAT)
- SAT modulo theories (SMT)
- (Mixed) integer linear programming (IP and MIP)
- Constraint programming (CP)
- ...
- Hybrid technologies

Techniques, *usually without* modelling and solvers:
- Dynamic programming (DP)
- Greedy algorithms
- Approximation algorithms
- Stochastic local search (SLS)
- Genetic algorithms (GA)
- ...
Constraint Programming Technology

Constraint programming (CP) offers methods & tools for:

what: Modelling constraint problems in a high-level language.

and

how: Solving constraint problems **intelligently** by:

- either default **search** upon pushing a button
- or **systematic search** guided by user-given strategies
- or **local search** guided by user-given (meta-)heuristics
- or **hybrid search**

plus inference, called **propagation**, but little **relaxation**.

**Slogan of CP:**

Constraint Program = Model [ + Search ]
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**General term:** balanced incomplete block design (BIBD).
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General term: **balanced incomplete block design (BIBD)**.
Example (BIBD integer model: $\checkmark \iff 1$ and $\neg \iff 0$)

1. int: nbrVarieties; int: nbrBlocks;
2. set of int: Varieties = 1..nbrVarieties;
3. set of int: Blocks = 1..nbrBlocks;
4. int: sampleSize; int: blockSize; int: balance;
5. array[Varieties,Blocks] of var 0..1: BIBD;
6. solve satisfy;
7. constraint forall(v in Varieties)
   (sampleSize = sum(BIBD[v,..]));
8. constraint forall(b in Blocks)
   (blockSize = sum(BIBD[..,b]));
9. constraint forall(v1, v2 in Varieties where v1 < v2)
   (balance = sum(b in Blocks) (BIBD[v1,b] * BIBD[v2,b]));

Example (Instance data for our AED)

1. nbrVarieties = 7; nbrBlocks = 7;
2. sampleSize = 3; blockSize = 3; balance = 1;
Reconsider the model fragment:

```plaintext
11 constraint forall(v1, v2 in Varieties where v1 < v2) 
12 (balance = sum(b in Blocks) (BIBD[v1,b]*BIBD[v2,b]));
```

This constraint is declarative (and by the way non-linear), so read it using only the verb “to be” or synonyms thereof:

For all two ordered varieties \(v_1\) and \(v_2\), the sum over all blocks \(b\) of the products \(\text{BIBD}[v_1,b]*\text{BIBD}[v_2,b]\) must equal \(\text{balance}\)

The constraint is not procedural:

For all two ordered varieties \(v_1\) and \(v_2\), we first add up, over all blocks \(b\), the products \(\text{BIBD}[v_1,b]*\text{BIBD}[v_2,b]\), and then we check whether that sum is equal to \(\text{balance}\)

The latter reading is appropriate for solution checking, but solution finding performs no such procedural summation.
Example (Idea for another BIBD model)

<table>
<thead>
<tr>
<th>Grain</th>
<th>Plots</th>
</tr>
</thead>
<tbody>
<tr>
<td>barley</td>
<td>{plot1, plot2, plot3}</td>
</tr>
<tr>
<td>corn</td>
<td>{plot1, plot4, plot5}</td>
</tr>
<tr>
<td>millet</td>
<td>{plot1, plot6, plot7}</td>
</tr>
<tr>
<td>oats</td>
<td>{plot2, plot4, plot6}</td>
</tr>
<tr>
<td>rye</td>
<td>{plot2, plot5, plot7}</td>
</tr>
<tr>
<td>spelt</td>
<td>{plot3, plot4, plot7}</td>
</tr>
<tr>
<td>wheat</td>
<td>{plot3, plot5, plot6}</td>
</tr>
</tbody>
</table>

**Constraints to be satisfied:**

1. Equal sample size: Every grain is grown in 3 plots.
2. Equal growth load: Every plot grows 3 grains.
3. Balance: Every grain pair is grown in 1 common plot.
Example (BIBD set model: a block set per variety)

1 ... 
2 ... 
3 ... 
4 ... 
5 array[Varieties] of var set of Blocks: BIBD; 
6 ... 
7 ... 
8 (sampleSize = card(BIBD[v]));
9 ... 
10 (blockSize = count(BIBD,b));
11 ... 
12 (balance = card(BIBD[v1] inter BIBD[v2]));

Example (Instance data for our AED)

1 ... 
2 ...
Example (Doctor rostering)

<table>
<thead>
<tr>
<th>Doctor</th>
<th>Mon</th>
<th>Tue</th>
<th>Wed</th>
<th>Thu</th>
<th>Fri</th>
<th>Sat</th>
<th>Sun</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>call</td>
<td>–</td>
<td>oper</td>
<td>–</td>
<td>oper</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>B</td>
<td>app</td>
<td>call</td>
<td>–</td>
<td>oper</td>
<td>–</td>
<td>–</td>
<td>call</td>
</tr>
<tr>
<td>C</td>
<td>oper</td>
<td>–</td>
<td>call</td>
<td>app</td>
<td>app</td>
<td>call</td>
<td>–</td>
</tr>
<tr>
<td>D</td>
<td>app</td>
<td>oper</td>
<td>–</td>
<td>call</td>
<td>oper</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>E</td>
<td>oper</td>
<td>–</td>
<td>oper</td>
<td>–</td>
<td>call</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

**Constraints to be satisfied:**

1. \#doctors-on-call / day = 1
2. \#operations / workday ≤ 2
3. \#operations / week ≥ 7
4. \#appointments / week ≥ 4
5. day off after operation day
6. . .

**Objective function to be minimised:**

- Cost: . . .
Example (Doctor rostering)

1 set of int: Days = 1..7;
2 set of int: Mon2Fri = 1..5;
3 enum: Doctors = {Dr A, Dr B, Dr C, Dr D, Dr E};
4 enum: ShiftTypes = {app, call, oper, none};
5
6 array[Doctors,Days] of var ShiftTypes: Roster;
7
8 solve minimize ...; % objective function
9
10 constraint forall(d in Days)
11    (count(Roster[..,d],call) = 1);
12 constraint forall(w in Mon2Fri)
13    (count(Roster[..,w],oper) <= 2);
14 constraint count(Roster,oper) >= 7;
15 constraint count(Roster,app) >= 4;
16 constraint forall(d in Doctors)
17    (regular(Roster[d,..], (oper none|app|call|none)*));
18 ... % other constraints
Example (Sudoku)

```plaintext
1 array[1..9,1..9] of var 1..9: Sudoku;
2 ... % load the hints
3 solve satisfy;
4 constraint forall(row in 1..9) (alldifferent(Sudoku[row, ..]));
5 constraint forall(col in 1..9) (alldifferent(Sudoku[.., col]));
6 constraint forall(i,j in {1,4,7}) (alldifferent(Sudoku[i..i+2, j..j+2]));
```
Using variables as indices: black magic?!

Example (Job allocation at minimal salary cost)

**Given** jobs 1..n and the salaries of workers 1..w, **find** a worker for each job, **such that** some constraints (on the qualifications of the workers for the jobs, on workload distribution, etc) are satisfied and the total salary cost is minimal:

1. array[1..w] of int: Salary;
2. array[1..n] of var 1..w: Worker; % job j by Worker[j]
3. solve minimize sum(j in 1..n)(Salary[Worker[j]]);
4. constraint ...; % qualifications, workload, etc

Example (Travelling salesperson over cities 1..n)

1. array[1..n,1..n] of float: Distance; % instance data
2. array[1..n] of var 1..n: Next; % go from c to Next[c]
3. solve minimize sum(c in 1..n)(Distance[c,Next[c]]);
4. constraint circuit(Next);
5. constraint ...; % side constraints, if any
Constraint-Based Modelling

The given models are expressed in a high-level constraint-based modelling language, typical of CP:

- There are several types for variables: integers (int), reals (float), Booleans (bool), strings (string), integer sets (set), and matrices thereof (array).

- There is a nice vocabulary of predicates (<, <=, =, !=, >=, >, alldifferent, circuit, regular, ...), functions (+, −, *, card, count, inter, sum, ...), and connectives (/\, \/, ...).

- There is support for both constraint satisfaction (satisfy) and constrained optimisation (minimize and maximize).

Most modelling languages are (much) lower-level than this!
Got the Moves, But Can’t Show It?!
The constraint predicates (\texttt{alldifferent, circuit, regular, \ldots}) and structured variable types (sets, \ldots) allow us both to model the structure of a constraint problem and to exploit that structure when solving it.

Dozens of constraint predicates (see the Catalogue) declaratively encapsulate complex propagation algorithms, including for $\sum_{i=1}^{n} a_i \cdot x_i \sim b$, where $\sim \in \{<, \leq, =, \neq, \geq, >\}$.

If the scope of a predicate is an unfixed number of variables (an array of variables, a set variable, or a string variable), then one speaks of a global constraint in a CP model.

There is no standardised CP modelling language: distinct CP solvers may support distinct predicates, possibly under distinct names and signatures, as well as distinct types.

\begin{itemize}
    \item But see the MiniZinc.org language & toolchain, which extends the spirit of AMPL and GAMS.
\end{itemize}
Pride:

*Constraint programming represents one of the closest approaches computer science has yet made to the Holy Grail of programming: the user states the problem, the computer solves it.*

— Eugene Freuder, a CP pioneer
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Prejudice:

The contribution of the article should be the reduction of an engineering problem to a known optimization format. […] showcases pseudo code […] submit this work to a journal interested in code semantics […].

— Reviewer of a paper of mine at a prestigious OR journal
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Correctness Is Not Enough for Models
Modelling is an Art!

There are good & bad models for each constraint problem:

- Different models of a problem may take different time on the same solver for the same instance.
- Different models of a problem may scale differently on the same solver for instances of growing size.

A tiny model change may accelerate the solving manyfold!

Good modellers are worth their weight in gold!

Use solvers, based on decades of cutting-edge research: you reuse hundreds of thousands of lines of highly tuned code that is very hard to beat on exact solving.
Outline

1. Constraint Problems
2. Constraint Programming Technology
3. CP Modelling
4. CP Solving
   - Systematic Search
   - Local Search
5. History of CP
6. Success Stories of CP
7. When (Not) to Use CP?
8. Bibliography
Constraint programming (CP) offers methods & tools for:

what: Modelling constraint problems in a high-level language.

and

how: Solving constraint problems intelligently by:

• either default search upon pushing a button
• or systematic search guided by user-given strategies
• or local search guided by user-given (meta-)heuristics
• or hybrid search

plus inference, called propagation, but little relaxation.

Slogan of CP:

Constraint Program = Model [ + Search ]
## Outline

1. Constraint Problems
2. Constraint Programming Technology
3. CP Modelling
4. **CP Solving**
   - Systematic Search
   - Local Search
5. History of CP
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CP Solving = Propagation + Search

A CP solver conducts search interleaved with propagation:

Each constraint has a propagator.
Propagation of one Constraint: Propagator

Example

Consider the constraint `CONNECTED([C_1, \ldots, C_n])`, which enforces max one stretch per colour among the \( n \) variables.

From

\[
\begin{array}{cccccccc}
\ldots & ? & \ldots & ? & ? & ? & \text{\textcolor{red}{\rule{2cm}{0.1pt}}} & \text{\textcolor{yellow}{\rule{2cm}{0.1pt}}} & \ldots & ? & \ldots
\end{array}
\]

the propagator of the `CONNECTED(\ldots)` constraint infers

\[
\begin{array}{cccccccc}
\text{\textcolor{red}{\rule{2cm}{0.1pt}}} & \text{\textcolor{black}{\rule{2cm}{0.1pt}}} & \text{\textcolor{red}{\rule{2cm}{0.1pt}}} & \text{\textcolor{red}{\rule{2cm}{0.1pt}}} & \text{\textcolor{red}{\rule{2cm}{0.1pt}}} & \text{\textcolor{yellow}{\rule{2cm}{0.1pt}}} & \text{\textcolor{red}{\rule{2cm}{0.1pt}}} & \text{\textcolor{yellow}{\rule{2cm}{0.1pt}}} & \ldots & \text{\textcolor{red}{\rule{2cm}{0.1pt}}} & \text{\textcolor{black}{\rule{2cm}{0.1pt}}} & \ldots
\end{array}
\]

Propagation is the elimination of the impossible values from the current domains of the variables, and thereby accelerates otherwise blind search.
Example

Consider the $n$-ary predicate `alldifferent`, with $n = 4$:

$$\text{alldifferent}([a, b, c, d])$$  \hspace{1cm} (1)

**Modelling:** (1) is equivalent to $\frac{n(n-1)}{2}$ binary constraints:

$$a \neq b \land a \neq c \land a \neq d \land b \neq c \land b \neq d \land c \neq d$$  \hspace{1cm} (2)

**Inference:** (1) propagates much better than (2). Example:

$$a \in \{4, 5\}, \ b \in \{4, 5\}, \ c \in \{3, \emptyset\}, \ d \in \{1, 2, \emptyset, \emptyset, 5\}$$

No propagation by (2).
Example

Consider the \( n \)-ary predicate \texttt{alldifferent}, with \( n = 4 \):

\[
\texttt{alldifferent}([a, b, c, d]) \tag{1}
\]

Modelling: (1) is equivalent to \( \frac{n(n-1)}{2} \) binary constraints:

\[
a \neq b \land a \neq c \land a \neq d \land b \neq c \land b \neq d \land c \neq d \tag{2}
\]

Inference: (1) propagates much better than (2). Example:

\[
a \in \{4, 5\}, \ b \in \{4, 5\}, \ c \in \{3, 4\}, \ d \in \{1, 2, 3, 4, 5\}
\]

No propagation by (2).
Example

Consider the $n$-ary predicate \texttt{alldifferent}, with $n = 4$:

\begin{equation}
\texttt{alldifferent}([a, b, c, d])
\end{equation}

Modelling: (1) is equivalent to $\frac{n(n-1)}{2}$ binary constraints:

\begin{equation}
a \neq b \land a \neq c \land a \neq d \land b \neq c \land b \neq d \land c \neq d
\end{equation}

Inference: (1) propagates much better than (2). Example:

\begin{align*}
a &\in \{4, 5\}, \quad b \in \{4, 5\}, \quad c \in \{3, 4\}, \quad d \in \{1, 2, 3, 4, 5\}
\end{align*}

No propagation by (2).
Example

Consider the \( n \)-ary predicate \texttt{alldifferent}, with \( n = 4 \):

\[
alldifferent ([a, b, c, d]) \quad (1)
\]

Modelling: \( (1) \) is equivalent to \( \frac{n(n-1)}{2} \) binary constraints:

\[
a \neq b \land a \neq c \land a \neq d \land b \neq c \land b \neq d \land c \neq d \quad (2)
\]

Inference: \( (1) \) propagates much better than \( (2) \). Example:

\[
a \in \{4, 5\}, \quad b \in \{4, 5\}, \quad c \in \{3, 4\}, \quad d \in \{1, 2, 3, 4, 5\}
\]

No propagation by \( (2) \). But perfect propagation by \( (1) \)!
Example

Consider the $n$-ary predicate \texttt{alldifferent}, with $n = 4$:

\[
\texttt{alldifferent}([a, b, c, d]) \quad (1)
\]

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\[
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\]

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\[
a \in \{4, 5\}, \; b \in \{4, 5\}, \; c \in \{3, 4\}, \; d \in \{1, 2, 3, 4, 5\}
\]

No propagation by (2). But perfect propagation by (1)!

**Search:** The \texttt{alldifferent} propagator is suspended, as its constraint currently does not surely hold.
Example

Consider the \( n \)-ary predicate \texttt{alldifferent}, with \( n = 4 \):

\[
\textsf{alldifferent}([a, b, c, d])
\]  

(1)

Modelling: (1) is equivalent to \( \frac{n(n-1)}{2} \) binary constraints:

\[
a \neq b \land a \neq c \land a \neq d \land b \neq c \land b \neq d \land c \neq d
\]  

(2)

Inference: (1) propagates much better than (2). Example:

\[
a \in \{4, 5\}, \quad b \in \{4, 5\}, \quad c \in \{3, 4\}, \quad d \in \{1, 2, 3, 4, 5\}
\]

No propagation by (2). But perfect propagation by (1)!

Search: The \texttt{alldifferent} propagator is suspended, as its constraint currently does not surely hold. If search or another propagator infers \( a = 4 \), then the \texttt{alldifferent} propagator is awakened: it infers \( b = 5 \) and is disposed of, as its constraint then surely holds, for any \( d \).
Example (Propagator for \textit{alldifferent})

Solutions to \texttt{alldifferent}([\(u, v, w, x, y, z\)]) map to maximum matchings in a bipartite graph for the domains:

\begin{itemize}
  \item \(u \in \{0, 1\}\)
  \item \(v \in \{1, 2\}\)
  \item \(w \in \{0, 2\}\)
  \item \(x \in \{1, 3\}\)
  \item \(y \in \{2, 3, 4, 5\}\)
  \item \(z \in \{5, 6\}\)
\end{itemize}
Example (Propagator for \texttt{alldifferent})

Mark all edges of \textit{some} maximum matching; Hopcroft-Karp algorithm takes $O(m\sqrt{n})$ time for $n$ variables and $m$ values:

\begin{align*}
  u & \in \{0, 1\} & u & \in \{0, 1\} \\
  v & \in \{1, 2\} & v & \in \{1, 2\} \\
  w & \in \{0, 2\} & w & \in \{0, 2\} \\
  x & \in \{1, 3\} & x & \in \{1, 3\} \\
  y & \in \{2, 3, 4, 5\} & y & \in \{2, 3, 4, 5\} \\
  z & \in \{5, 6\} & z & \in \{5, 6\}
\end{align*}
Example (Propagator for **alldifferent**)

Mark all edges of *some* maximum matching; Hopcroft-Karp algorithm takes $O(m\sqrt{n})$ time for $n$ variables and $m$ values:

- $u \in \{0, 1\}$
- $v \in \{1, 2\}$
- $w \in \{0, 2\}$
- $x \in \{1, 3\}$
- $y \in \{2, 3, 4, 5\}$
- $z \in \{5, 6\}$
Example (Propagator for \texttt{alldifferent})

Mark all other edges in \textit{all} other maximum matchings, exploiting a result by J. Petersen in \textit{Acta Mathematica} 1891:

\[\begin{align*}
    u & \in \{0, 1\} \\
v & \in \{1, 2\} \\
w & \in \{0, 2\} \\
x & \in \{1, 3\} \\
y & \in \{2, 3, 4, 5\} \\
z & \in \{5, 6\}
\end{align*}\]
Example (Propagator for \texttt{alldifferent})

Mark all other edges in \textit{all} other maximum matchings, exploiting a result by J. Petersen in \textit{Acta Mathematica} 1891:

\[
\begin{align*}
  u &\in \{0, 1\} & \quad u \\
v &\in \{1, 2\} & \quad v \\
w &\in \{0, 2\} & \quad w \\
x &\in \{1, 3\} & \quad x \\
y &\in \{2, 3, 4, 5\} & \quad y \\
z &\in \{5, 6\} & \quad z
\end{align*}
\]
Example (Propagator for \textit{alldifferent})

Every still unmarked edge is in \textit{no} maximum matching. Propagate accordingly within the current domains:

- \( u \in \{0, 1\} \)
- \( v \in \{1, 2\} \)
- \( w \in \{0, 2\} \)
- \( x \in \{1, 3\} \)
- \( y \in \{2, 3, 4, 5\} \)
- \( z \in \{5, 6\} \)
Example (Propagator for \textit{alldifferent})

Every still unmarked edge is in \textit{no} maximum matching. Propagate accordingly within the current domains:

\begin{align*}
  u &\in \{0, 1\} \\
  v &\in \{1, 2\} \\
  w &\in \{0, 2\} \\
  x &\in \{1, 3\} \\
  y &\in \{2, 4, 5\} \\
  z &\in \{5, 6\}
\end{align*}

\begin{itemize}
  \item $u \in \{0, 1\}$
  \item $v \in \{1, 2\}$
  \item $w \in \{0, 2\}$
  \item $x \in \{1, 3\}$
  \item $y \in \{2, 4, 5\}$
  \item $z \in \{5, 6\}$
\end{itemize}
Search + Propagation of All Constraints

Example (BIBD: AED partial assignment)

<table>
<thead>
<tr>
<th></th>
<th>plot1</th>
<th>plot2</th>
<th>plot3</th>
<th>plot4</th>
<th>plot5</th>
<th>plot6</th>
<th>plot7</th>
</tr>
</thead>
<tbody>
<tr>
<td>barley</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>corn</td>
<td>✓</td>
<td>–</td>
<td>–</td>
<td>✓</td>
<td>✓</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>millet</td>
<td>✓</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>oats</td>
<td>–</td>
<td>✓</td>
<td>–</td>
<td>✓</td>
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<td>✓</td>
<td>–</td>
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<tr>
<td>rye</td>
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<td>?</td>
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<tr>
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<td></td>
<td></td>
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Search + Propagation of All Constraints

Example (BIBD: AED partial assignment)

<table>
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<tr>
<th></th>
<th>plot1</th>
<th>plot2</th>
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<tr>
<td>barley</td>
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<td>–</td>
</tr>
<tr>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

But plot1 cannot grow rye as that would violate the second constraint (every plot grows 3 grains).
Search + Propagation of All Constraints

Example (BIBD: AED partial assignment)

<table>
<thead>
<tr>
<th></th>
<th>plot1</th>
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<td>oats</td>
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</table>

But plot1 **cannot** grow rye as that would violate the second constraint (every plot grows 3 grains).
### Search + Propagation of All Constraints

**Example (BIBD: AED partial assignment)**

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<tr>
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But plot1 **cannot** grow rye as that would violate the second constraint (every plot grows 3 grains). Actually, plot1 **cannot** grow oats, spelt, or wheat either, for the same reason, and this was **already propagated** when trying the search guess that plot1 grow millet!
Search + Propagation of All Constraints

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**Guess:** Let plot2 grow rye. *(Strategy: ✓ guesses first.)*
Example (BIBD: AED partial assignment)

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Guess: Let plot2 grow rye. (Strategy: ✓ guesses first.)
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**Propagation:** plot2 cannot grow spelt and wheat as otherwise the second constraint (every plot grows 3 grains) would be violated for plot2.
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**Propagation:** plot2 cannot grow spelt and wheat as otherwise the second constraint (every plot grows 3 grains) would be violated for plot2.
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**Propagation:** plot3, plot4, and plot6 **cannot** grow rye as otherwise the third constraint (every grain pair is grown in 1 common plot) would be violated.
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**Propagation:** plot5 and plot7 must grow rye as otherwise the first constraint (every grain is grown in 3 plots) would be violated for rye.
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**Propagation:** plot5 and plot7 must grow rye as otherwise the first constraint (every grain is grown in 3 plots) would be violated for rye.
### Example (BIBD: AED partial assignment)

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<tr>
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**Propagation:** plot3 must grow spelt and wheat as otherwise the second constraint (every plot grows 3 grains) would be violated for plot3.
### Example (BIBD: AED partial assignment)

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**Propagation:** plot3 must grow spelt and wheat as otherwise the second constraint (every plot grows 3 grains) would be violated for plot3.
### Example (BIBD: AED partial assignment)

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**Common fixpoint reached:** No more propagation possible.
**Example (BIBD: AED partial assignment)**

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</tbody>
</table>

**Guess:** Let plot4 grow spelt. (Strategy: ✓ guesses first.)

**Propagation:** etc.
Systematic search, for a satisfaction problem:

1: propagate all constraints; backtrack if empty domain
2: if only fixed variables, then show solution & backtrack
3: while there is at least one suspended propagator do
4: select unfixed variable, \( v \), of current domain \( \text{dom}(v) \)
5: partition \( \text{dom}(v) \) using guesses (say \( v = d \) & \( v \neq d \), or \( v > d \) & \( v \leq d \), for a selected value \( d \in \text{dom}(v) \))
6: for each guess: recurse upon adding it as constraint

For an optimisation problem: before backtracking at line 2 add the constraint that any next solution must be better.

**Strategies:**

- Line 4: variable selection strategy: smallest domain, ...
- Line 5: value selection strategy: maximum, median, ...
- Line 5: guess selection strategy: equality, bisection, ...
- Tree exploration: depth-first search, ...

Example (Search for the BIBD integer model)

```plaintext
solve :: int_search(BIBD,input_order,indomain_max) satisfy;
```
### Example (BIBD: AED assignment after $i$ moves)

<table>
<thead>
<tr>
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</tr>
</tbody>
</table>

1. **Equal sample size**: Every grain is grown in 3 plots. Satisfied at initialisation and by each move: invariant.

2. **Equal growth load**: Every plot grows 3 grains. Currently satisfied: zero violation.

3. **Balance**: Every grain pair is grown in 1 common plot. But, e.g., oats & rye are grown in $2 > 1$ common plots.
Example (BIBD: AED assignment after $i$ moves)

<table>
<thead>
<tr>
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</table>

1. Equal sample size: Every grain is grown in 3 plots. Satisfied at initialisation and by each move: invariant.


3. Balance: Every grain pair is grown in 1 common plot. But, e.g., oats & rye are grown in 2 > 1 common plots.

Selected move: let plot6 instead of plot5 grow oats.
### Example (BIBD: AED assignment after $i$ moves)

<table>
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1. Equal sample size: Every grain is grown in 3 plots. Satisfied at initialisation and by each move: invariant.
3. Balance: Every grain pair is grown in 1 common plot. But, e.g., oats & rye are grown in $2 > 1$ common plots.

**Selected move:** let plot6 instead of plot5 grow oats.
### Example (BIBD: AED assignment after $i + 1$ moves)

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1. **Equal sample size**: Every grain is grown in 3 plots. Satisfied at initialisation and by each move: invariant.
2. **Equal growth load**: Every plot grows 3 grains. But plot5 grows $2 < 3$ grains; plot6 grows $4 > 3$ grains.
3. **Balance**: Every grain pair is grown in 1 common plot. But, e.g., corn & oats are grown in $2 > 1$ common plots.
Example (BIBD: AED assignment after $i + 1$ moves)

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1. Equal sample size: Every grain is grown in 3 plots. Satisfied at initialisation and by each move: invariant.

2. Equal growth load: Every plot grows 3 grains. But plot5 grows $2 < 3$ grains; plot6 grows $4 > 3$ grains.

3. Balance: Every grain pair is grown in 1 common plot. But, e.g., corn & oats are grown in $2 > 1$ common plots.

Selected move: let plot5 instead of plot6 grow corn.
Example (BIBD: AED assignment after $i + 1$ moves)

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</tr>
<tr>
<td>spelt</td>
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<td>✓</td>
<td>✓</td>
<td>−</td>
<td>−</td>
<td>✓</td>
</tr>
<tr>
<td>wheat</td>
<td>−</td>
<td>−</td>
<td>✓</td>
<td>−</td>
<td>✓</td>
<td>✓</td>
<td>−</td>
</tr>
</tbody>
</table>

1. Equal sample size: Every grain is grown in 3 plots. Satisfied at initialisation and by each move: invariant.

2. Equal growth load: Every plot grows 3 grains. But plot5 grows $2 < 3$ grains; plot6 grows $4 > 3$ grains.

3. Balance: Every grain pair is grown in 1 common plot. But, e.g., corn & oats are grown in $2 > 1$ common plots.

Selected move: let plot5 instead of plot6 grow corn.
### Example (BIBD: AED assignment after $i + 2$ moves)

<table>
<thead>
<tr>
<th></th>
<th>plot1</th>
<th>plot2</th>
<th>plot3</th>
<th>plot4</th>
<th>plot5</th>
<th>plot6</th>
<th>plot7</th>
</tr>
</thead>
<tbody>
<tr>
<td>barley</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>corn</td>
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<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
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<tr>
<td>millet</td>
<td>✓</td>
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<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>oats</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
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</tr>
<tr>
<td>rye</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
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<td>spelt</td>
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<tr>
<td>wheat</td>
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<td>✓</td>
</tr>
</tbody>
</table>

1. **Equal sample size**: Every grain is grown in 3 plots. Satisfied at initialisation and by each move: invariant.

2. **Equal growth load**: Every plot grows 3 grains. Currently satisfied: zero violation.

3. **Balance**: Every grain pair is grown in 1 common plot. Currently satisfied: zero violation.

**Stop search**: All constraints are satisfied.
Local search:

1. let $s$ and $s^*$ be the same computed initial assignment
2. while there are violated constraints & iterations left do
3. select a move on $s$; let $s'$ be the reached assignment
4. if $s'$ is better than $s^*$ then $s^* := s'$
5. $s := s'$
6. return $s^*$

Heuristics: What move to select?

- Line 3: assign, flip, swap, add, drop, transfer, …
- Line 3: best / first / random improvement, …

Meta-heuristics: How to escape local optima?

- Lines 2 to 5: simulated annealing, tabu search, …
Heuristics are an Art!

There are good & bad heuristics for each problem model:

- Different heuristics for a model may take different time on the same solver for the same instance.

- Different heuristics for a model may scale differently on the same solver for instances of growing size.

A tiny heuristic tweak may accelerate the solving manyfold!

Good heuristicians are worth their weight in gold!

Use solvers, based on decades of cutting-edge research: you reuse hundreds of thousands of lines of highly tuned code that is very hard to beat.
Outline

1 Constraint Problems
2 Constraint Programming Technology
3 CP Modelling
4 CP Solving
   • Systematic Search
   • Local Search
5 History of CP
6 Success Stories of CP
7 When (Not) to Use CP?
8 Bibliography
Stand-alone languages:

- **ALICE** by Jean-Louis Laurière, France, 1976
- **CHIP** at ECRC, Germany, 1987 – 1990, then marketed by Cosytec, France
- **OPL**, by P. Van Hentenryck, USA, and ILOG, France: front-end to both **ILOG CP Optimizer** and **ILOG CPLEX**
- **Comet**, by P. Van Hentenryck and L. Michel, USA
- **MiniZinc**, at U. of Melbourne and Monash U., Australia
- 

Libraries (the ones listed before “;” are open-source):

- **Prolog**: ECLiPSe, ...; SICStus Prolog, ...
- **C++**: Gecode, OR-Tools; IBM CP Optimizer, CHIP, ...
- **Java**: Choco, Google OR-Tools, JaCoP, ...; ...
- **Scala**: OscaR; ...
- 

STD
Outline

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Success Stories by CP Users and Contributors:

Success stories: CP = technology of choice in scheduling, configuration, personnel rostering, timetabling, . . .
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Scope of Constraint Programming

CP has a wide scope, as it addresses:

- satisfaction problems and optimisation problems
- discrete variables and continuous variables
- linear constraints and non-linear constraints

in principle in any combinations thereof, by:

- systematic search, if optimality more crucial than speed
- local search, if speed is more crucial than optimality
Common Misconceptions about CP

- CP is for experts: the predicate vocabulary is large.
  But there are tools helping you identify predicates, and about a dozen predicates take you a very long way!

- CP is for experts: the search strategy is mandatory.
  No, it is optional: there is adaptive autonomous search!

- CP’ers claim CP is a silver bullet for NP-hard problems.
  No: CP solvers are complementary in strength to MIP, SAT, SMT, . . . solvers and to local search, which leads to hybrid optimisation technologies: LCG, . . . !
Bluffer’s Guide to the C* Alphabet Soup

- CP solvers = constraint solvers
  MIP, SAT, ... solvers are also constraint solvers, whether known as such in those communities or not!

- CP = CLP = constraint logic programming
  Many modern CP solvers are not Prolog libraries!

  CLP(Q, R) solvers ⊂ CLP solvers ⊂ CP solvers

CP is a granddaughter of logic programming (LP)!

- CP = CSP = constraint satisfaction programming
  CSP = constraint satisfaction problem, an AI term.
  MIP, SAT, ... solvers also solve constraint problems, whether those communities use that term or not!
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Opportunities for CP

Rapid prototyping, with high solving performance, when:
- Constraints are, still or again, subject to experiments
- Partition into hard & soft constraints yet undetermined

Combinatorial structure is impure, due to side constraints.
It is time to consider all or more problem constraints.
Domain knowledge exploitable for problem-specific search.
It is a configuration problem.
It is a personnel rostering problem.
It is a scheduling problem.
It is a time-tabling problem.


M. Milano, P. Van Hentenryck, et al.  
The future of constraint programming.  
*Constraints*, special issue, 19(2), 2014.

M. Wallace.  
Constraint programming – The paradigm to watch.  

Ph. Baptiste, C. Le Pape, and W. Nuijten.  
Constraint-Based Scheduling.  

P. Van Hentenryck and L. Michel.  
Constraint-Based Local Search.  