Learning: from CP to SAT and back again

Topics in this Series

- Why SAT & Constraints?
- SAT basics
- Constraints basics
- Encodings between SAT and Constraints
- Watched Literals in SAT and Constraints
- Learning in SAT and Constraints
- Lazy Clause Generation + SAT Modulo Theories

Learning

- *Not* talking about classical machine learning
  - though it probably falls within that definition
  - and ML has interesting applications in constraints ...
  - but still, not talking about that
- Talking about learning during search
  - parts of the search space that are no good
  - learnt at large cost
  - can be avoided in future at low cost

Obvious advantage

- Search is exponential
- subsearches are exponential
  - and tell us facts that were expensive to find out
  - so let's *learn* those facts
  - and deduce them and similar facts faster in the rest of search

Obvious problem

- How do we learn facts?
  - And reuse them in the future
- We’re never going to revisit the identical search state ever again
  - so we have to *abstract* what we have learnt
  - so how do we work out something general from the specifics of this case?
  - and work out how to apply it elsewhere?
  - and with good cost-benefit ratio?

Learning in SAT & Constraints

- From Constraints ...
  - Conflict directed backjumping
- ... to SAT
  - Learning in SAT
  - VSIDS
- ... and back again
  - s-learning and g-learning in Constraints

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Learning & Backjumping in Constraints

- In this area Constraints seems to have a longer history
- Backjumping
  - Gaschnig 1977
- Learning
  - Dechter & Frost, 1990, 1994
- But I'm going to start with Conflict-directed backjumping
  - Prosser, 1993

CBJ

- Sometimes say "backjumping" as generic term
  - but dangerous as BJ is a specific algorithm (and not as good)
- Conflict directed backjumping
  - CBJ
  - I will use CBJ to include variants like FC-CBJ, MAC-CBJ
  - Patrick Prosser, 1993
    - 617 citations as I write (Google Scholar)
    - compare 215 for my most cited paper

Conflict Directed Backjumping

- Usual to distinguish between learning and backjumping based approaches
- CBJ
  - we avoid backtracking to any node
    - which is above the current node
    - and where the opposite branching choice ***cannot*** help
    - because we can ***prove*** it will not
- Learning
  - we reuse the information learnt at this node
  - at nodes which are not ancestors of the current node

Conflict set

- Key idea in CBJ
  - Same as explanation coming later
    - but I'll use the CBJ word here
- A **conflict set** at a failed node
  - is a set of assigned variables such that
  - if **any other** variable is changed there is no solution
  - equivalently:
    - every assignment with the current values in the conflict set, and arbitrary values elsewhere
    - is not a solution

Conflict Set Example

- x in {1,2,3}
- y assigned to 1
- **x < y**
  - conflict set is {y}
  - there is no possible value of x
  - no matter how many other variables there are in the problem

CBJ algorithm

- Whenever we get a failure, compute conflict set
- Jump back [i.e. backtrack to...] the most recently assigned variable *x* in conflict set
- Discard any search nodes between current node and *x*
- *Merge* current c.s. with existing c.s. of *x*
- *If any remaining values of x*
  - try next value
  - *else* repeat this slide

Computing Conflict Sets

- Two cases
  - backtracking
  - propagating
- Quick summary...
  - *every propagation always has a conflict set*
  - *merging is taking the union*

Merging Conflict Sets on Backtracking

- Say we have tried $x = a, b, c$
  - and we have cs_{abc} *(merged conflict set for x)*
  - and value $x = d$ has just failed
    - with cs_d which must have x in it
      - *why must x be in it?*

Merging Conflict Sets on Backtracking

- Say we have tried $x = a, b, c$
  - and we have cs_{abc} *(merged conflict set for x)*
  - and value $x = d$ has just failed
    - with cs_d which must have x in it
- How do we merge cs_{abc} and cs_d ?
  - Simply take $cs_{abc} \cup cs_d - \{x\}$
  - *why?*

$cs_{abc} \cup cs_d - \{x\}$

- When [if] we ever backjump from *x*
  - we need to know every variable which changing could lead to a solution
  - We need everything in cs_{abc}
    - otherwise $x = a$, $x=b$ or $x=c$ might work
  - And everything in cs_d
    - otherwise $x=d$ might work
  - But not *x* because we are backjumping from *x*

Conflict sets & Propagation

- If we are propagating (we always are)
- We can’t ignore propagation for c.s’s
- **x < y, y < z**
  - x in {1,2,3}, y in {1,2,3}, z in {2,3}
    - z assigned to 2 at search node
    - Then y assigned to 1,
    - then x fails with c.s = {y}
  - So we backjump to last var in c.s., that is *y*
    - *but there is no such node so we fail*
  - But we should backjump to z=2
    - then try z=3 and we can carry on

Conflict sets & Propagation

- Every time a possible value *x=a* is deleted
  - we record a conflict set for the deletion
- a c.s. for deletion is just like a failure c.s.
  - set so that if the variables in it take their current values, then x=a is impossible
- When we propagate, merge c.s.'s which played role
  - e.g. if we are doing AC
  - relevant c.s.'s are deleted values in constraint which otherwise would form a support for *x=a*

Conflict sets & Propagation

- **x < y, y < z**
  - x in {1,2,3}, y in {1,2,3}, z in {2,3}
    - z assigned to 2 at search node
    - Then *y=2 is deleted,* conflict set {z}
    - And *y=3* is deleted, conflict set {z}
    - then x fails with c.s = merge(y=1/{z},y=2/{z}) = {z}
  - So we backjump to last var in c.s., i.e. *z*
    - then try z=3 and we can carry on

Conflict sets & explanations

- Going to return in a bit to key questions:
  - how do we compute explanations (conflict sets) from propagators?
  - and then handle the merging of them from propagators?

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CBJ in SAT

- Very natural view of CBJ in SAT
- conflict sets are *clauses*
    - *e.g. conflict set after failure of x=a is {y,z}, with y=b, z=c*
    - *-xa OR -yb OR -yc*
    - *e.g. conflict set for x=b is {y,w} with w=d*
    - *-xb OR -yb OR -wd*
- conflict set merging is *resolution*
    - We have At-Least-One clause
    - *xa OR xb*
        - resolve to get *xb OR -yb OR -zc*
        - resolve to get *-yb OR -zc OR -wd*

CBJ in SAT

- *Very easy indeed* to work out conflict set
- If failure (i.e. empty clause) arises in clause C ...
  - ... conflict set is C
- If clause C becomes unit setting $x=0$
  - ... conflict set explaining $x \mathrel{!}= 1$ is C

CBJ in SAT

- CBJ was brought across to SAT in 96, 97
- "Using CSP Look-Back Techniques to Solve Real-World SAT Instances", 1997
  - Bayardo & Schrag
    - 593 citations (Google Scholar)
- All SAT solvers do backjumping/learning
  - Much research on SAT solvers in mid-late 90s
  - Bayardo & Schrag porting of CBJ one key piece of research

CBJ in SAT

- Another key piece of work was GRASP
  - Marques-Silva & Sakallah, 97, 99
    - 693 and 811 citations for the two papers
  - Doesn't cite Prosser this time
    - Ginsberg 93 (529 citations)
    - Sussmann/Stallmann 77 (677 citations)
    - i.e. still brought backjumping to SAT from Constraints

Citation numbers

- 617 (Prosser),
- 677 (Sussmann & Stallmann)
- 529 (Ginsberg)
- 593 (Bayardo & Schrag)
- 693 (Marques-Silva & Sakallah)
- 811 (Marques-Silva & Sakallah)
- To get an idea of scale
  - 656 citations
  - Jean-Charles Régin 1994, All-different GAC propagator

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Learning in SAT

- Going to present this with constraints in mind
  - so sometimes slightly more general than what SAT does
- But still will start with SAT
  - and move on to constriants later

Explanation (general)

- An *explanation* for a assignment ($x = a$) or disassignment ($x \mathrel{!}= a$) is
  - a set of assignments or disassignments
  - such that if this set is all (dis-)assigned
    - appropriate propagation level
    - will force $x = a$ (or $x\mathrel{!}=a$)

Explanation (Unit propagation)

- An *explanation* for a literal (*x = 0 or x = 1)* is
  - a set of literals
  - such that if this set is all assigned
    - unit propagation
    - will force the literal to be true
    - i.e. the negation of remaining literals in clause which caused the unit propagation to happen
- Also *explanation* for *failure*
    - exactly analogous
    - i.e. negation of all literals in failed clause

SAT Example

- You must invite somebody:
- The ambassador asks you to invite a Francophone ambassador so his daughter can practice her French:
- The Belgian, German and Dutch ambassadors are badly behaved when they get together, so they mustn't all be invited:
- If you invite the Dutch ambassador, you must also invite the Belgian ambassador:

- $B \lor N \lor F \lor G$
- $B \lor F$
- $\neg B \lor \neg G \lor \neg N$.
- $N \Rightarrow B$
  - $\neg N \lor B$

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- $B \vee N \vee F \vee G$
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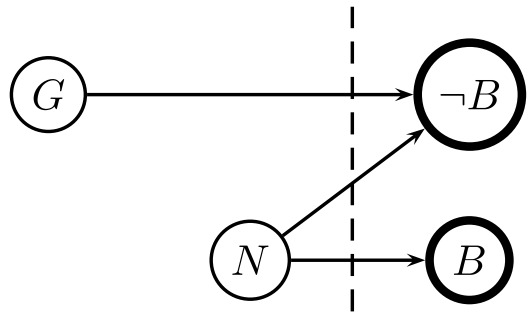
1. $B \vee N \vee F \vee G$
2. $B \vee F$
3. $\neg B \vee \neg G \vee \neg N$.
4. $\neg N \vee B$

- Suppose we have N, G
  - (4) explanation of B is N
  - (3) explanation of ¬B is G, N
  - (2) explanation of F is ¬B

SAT Implication Graph

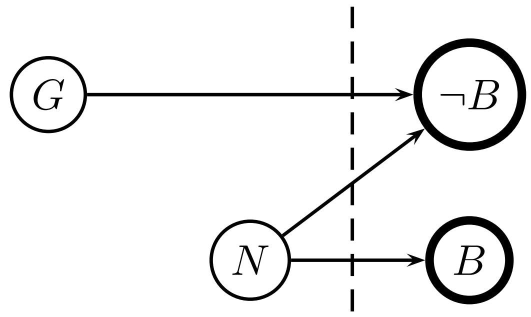
- An implication graph (IG) for the current state of variables is a directed acyclic graph where
  - each node is a currently true literal, e.g. v when variable v ← 1, and
  - there is an edge from u to v iff u appears in the explanation for v.



- $B \vee N \vee F \vee G$
- $B \vee F$
- $\neg B \vee \neg G \vee \neg N$.
- $\neg N \vee B$
- set **G** and **N**

Implication Graph Cut

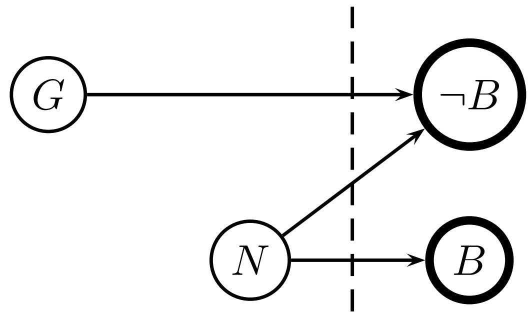
- A **cut** of an IG containing mutually inconsistent nodes is a partition (S,T) of vertices such that
  - all nodes corresponding to decision assignments belong to S,
  - the mutually inconsistent nodes are in T
  - if a node x ∈ T,
    - **either** all its direct predecessors in T
    - **or** all its direct predecessors are in S.
- Asserting all events of a cut and enforcing the same consistency level as the explanations were built with will lead to the same failure.
- Cuts often written by literals immediately to left of cut
  - e.g. {G, N}



- B ∨ N ∨ F ∨ G
- B ∨ F
- ¬B ∨ ¬G ∨ ¬N.
- ¬N ∨ B
- set **G** and **N**

Cuts in graphs

- A cut in the graph gives us something we can learn
- We can add a clause from the cut
  - e.g. cut {G,N}
  - learn ¬G ∨ ¬N
  - Should avoid the same mistake in the future
- But how do we find a good cut?

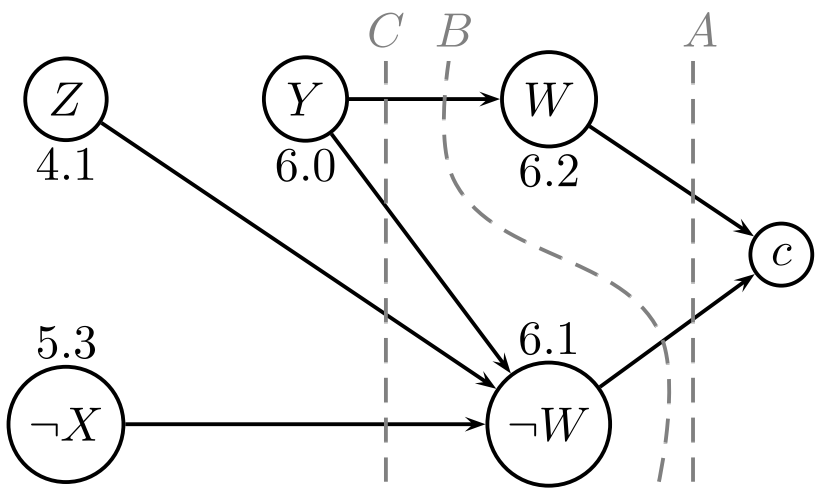


- B ∨ N ∨ F ∨ G
- B ∨ F
- ¬B ∨ ¬G ∨ ¬N.
- ¬N ∨ B
- set **G** and **N**

First UIP Cut

- If cut is too specific...
  - doesn't actually avoid the work of branching
- If cut is too general...
  - it may not save work
  - e.g. just the set of branching decisions
- Want a compromise
  - First Unique Implication Point
  - Find cut such that both contradictory literals forced by branching decision and nothing else at this depth

- Clauses (fragment)
  - $\neg Y \lor W$
  - $X \lor \neg Y \lor \neg Z \lor \neg W$



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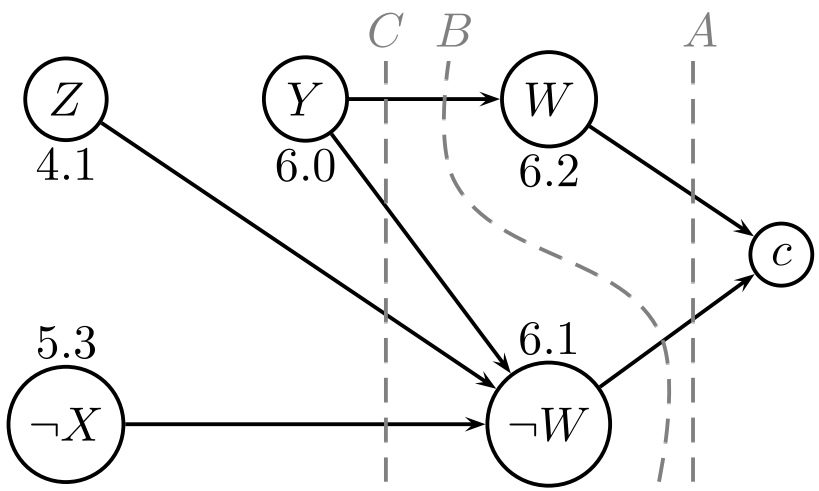

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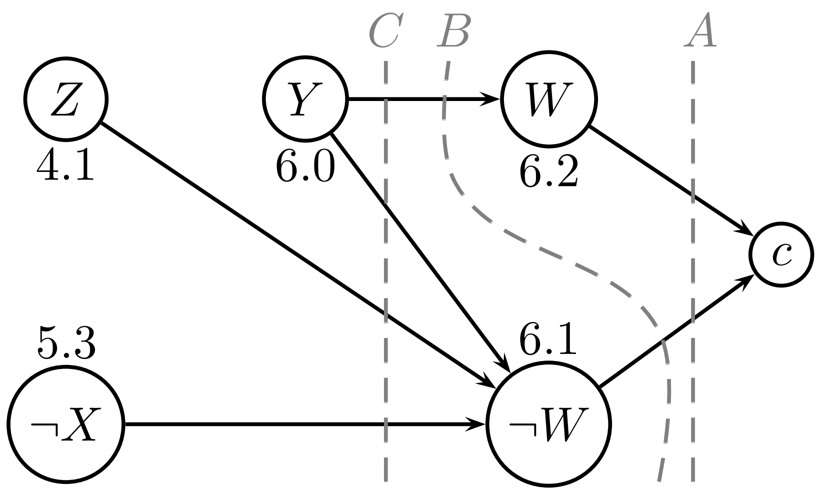
First UIP Cut

- Depth annotations
  - *6.0 = set at depth 6 by search*
  - *4.1 = set at depth 4, first propagation*
- *c* represents conflict
  - i.e. empty clause
  - i.e. ¬Y ∨ W with Y=1, W=0

- Clauses (fragment)
  - ¬Y ∨ W
  - X ∨ ¬Y ∨ ¬Z ∨ ¬W



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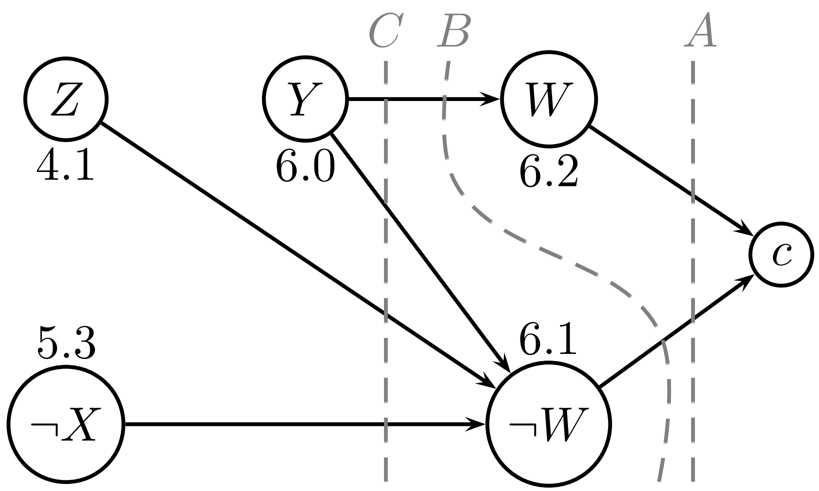
First UIP Cut Algorithm

- Start with T = trivial cut of conflict
- Loop until we have First UIP cut
  - choose deepest node N with predecessors in S
  - add all predecessors of N to T

- Clauses (fragment)
  - ¬Y ∨ W
  - X ∨ ¬Y ∨ ¬Z ∨ ¬W



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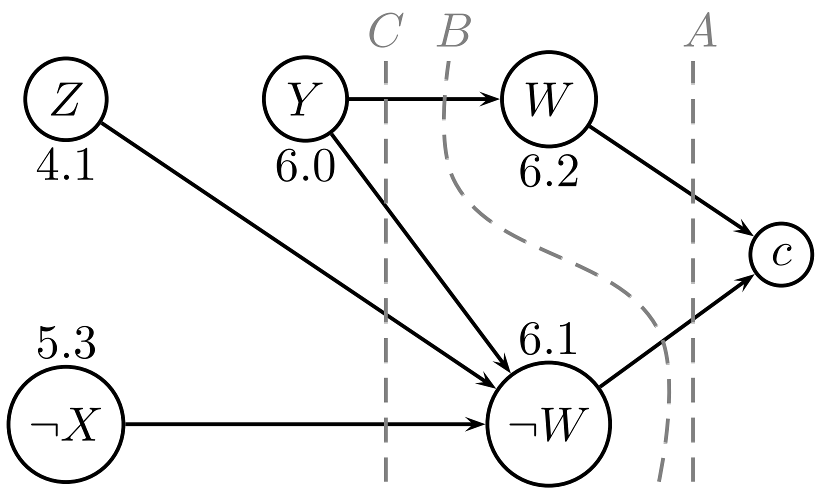
First UIP Cut Algorithm

- Start with T = trivial cut of conflict *c*
- Loop until we have First UIP cut
  - choose deepest node N with predecessors in S
  - add all predecessors of N to T
  - Remove N from T
- Theorem:
  - algorithm always gives us first UIP cut

- Clauses (fragment)
  - ¬Y ∨ W
  - X ∨ ¬Y ∨ ¬Z ∨ ¬W



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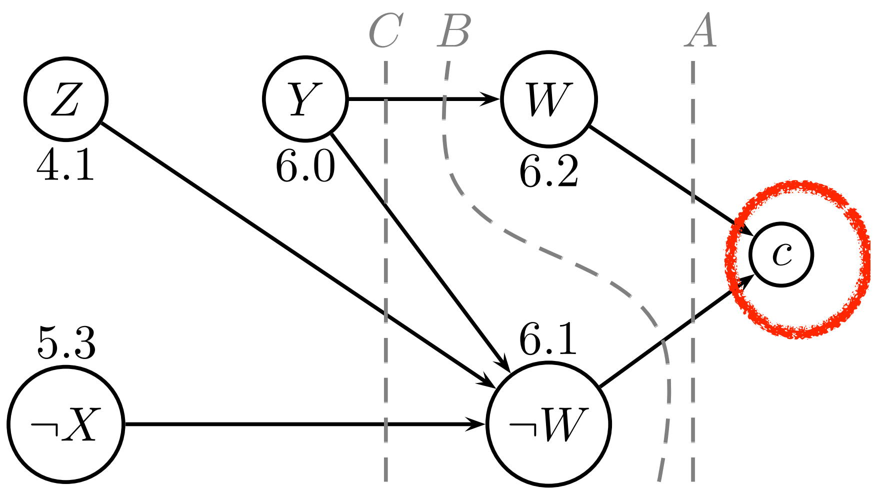
First UIP Cut Algorithm

- Start with T = trivial cut of conflict
  - *T* = {*c*}
    - line A
  - deepest node in T = *c*

- Clauses (fragment)
  - ¬Y ∨ W
  - X ∨ ¬Y ∨ ¬Z ∨ ¬W



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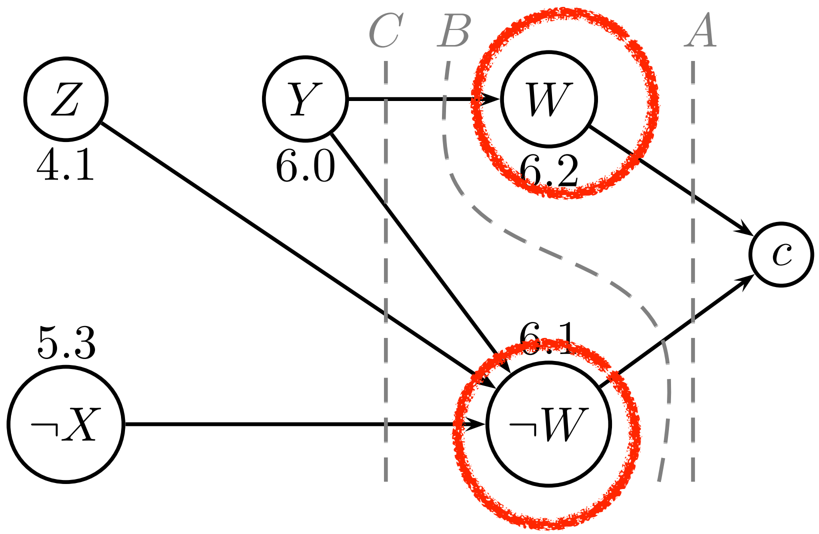
First UIP Cut Algorithm

- Start with T = trivial cut of conflict
- *T* = {*c*} *(line A)*
- deepest node in T = *c*
  - Add predecessors
  - Add W, ¬W to T
  - Remove *c*

- Clauses (fragment)
  - ¬Y ∨ W
  - X ∨ ¬Y ∨ ¬Z ∨ ¬W



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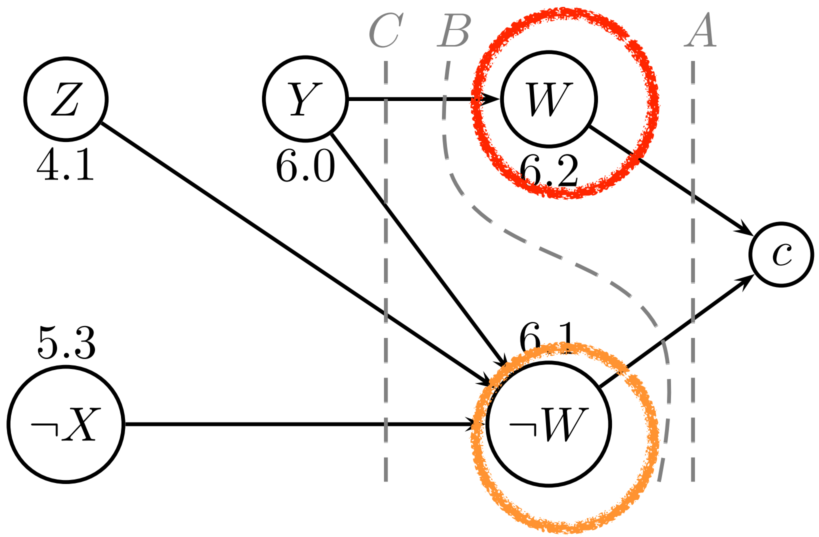
First UIP Cut Algorithm

- Add *W*, ¬W to T
    - Depths 6.1, 6.2
    - T = {*W*, ¬*W*}
        - *line B*
- deepest node in T = *W*
    - Add predecessors
        - Add *Y* to T
    - Remove *W*

- Clauses (fragment)
    - ¬Y ∨ W
    - X ∨ ¬Y ∨ ¬Z ∨ ¬W



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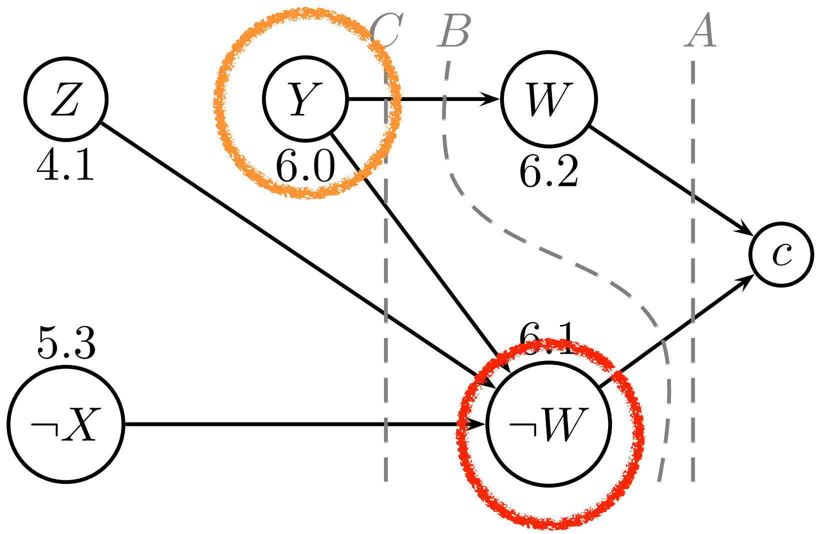
First UIP Cut Algorithm

- Add *Y* to T
  - T = {*Y*, ¬*W*}
  - *Depths 6.0, 6.1*
- deepest node in T = ¬*W*
  - Add predecessors
    - Add *Z*, ¬*X*, *Y* to T
  - Remove ¬*W*

- Clauses (fragment)
  - ¬Y ∨ W
  - X ∨ ¬Y ∨ ¬Z ∨ ¬W



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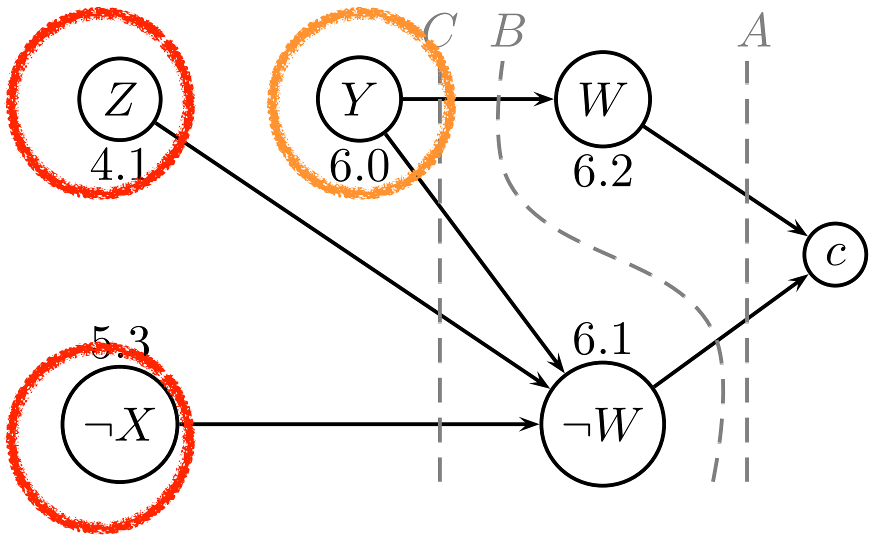
First UIP Cut Algorithm

- Add Z, ¬X, Y to T
  - T = {Y, Z, ¬X}
    - *line C*
  - *Depths 6.0, 4.1, 5.3*

- Clauses (fragment)
  - ¬Y ∨ W
  - X ∨ ¬Y ∨ ¬Z ∨ ¬W



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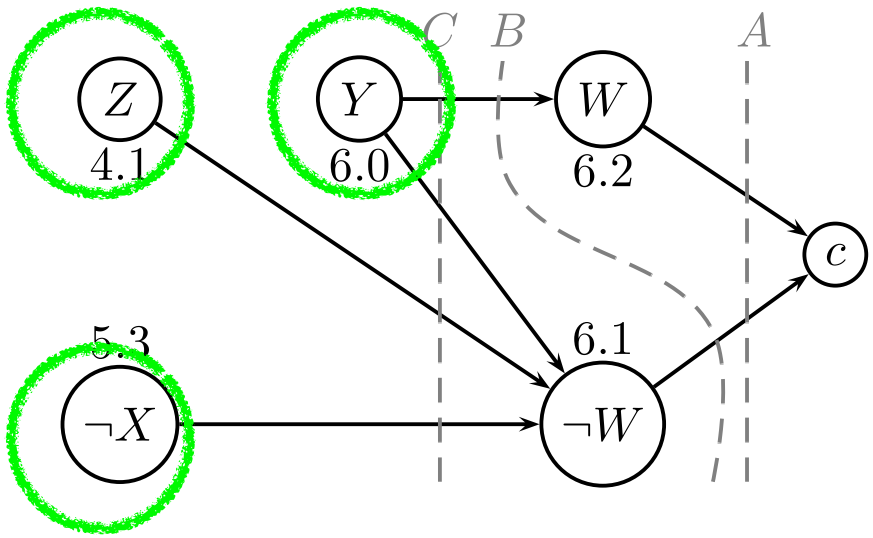
First UIP Cut Algorithm

- STOP
  - We have reached UIP
  - Unique depth 6 node
    - i.e. search decision
  - T is firstUIP Cut
- Cut is T = {Y, Z, ¬X}

- Clauses (fragment)
  - ¬Y ∨ W
  - X ∨ ¬Y ∨ ¬Z ∨ ¬W



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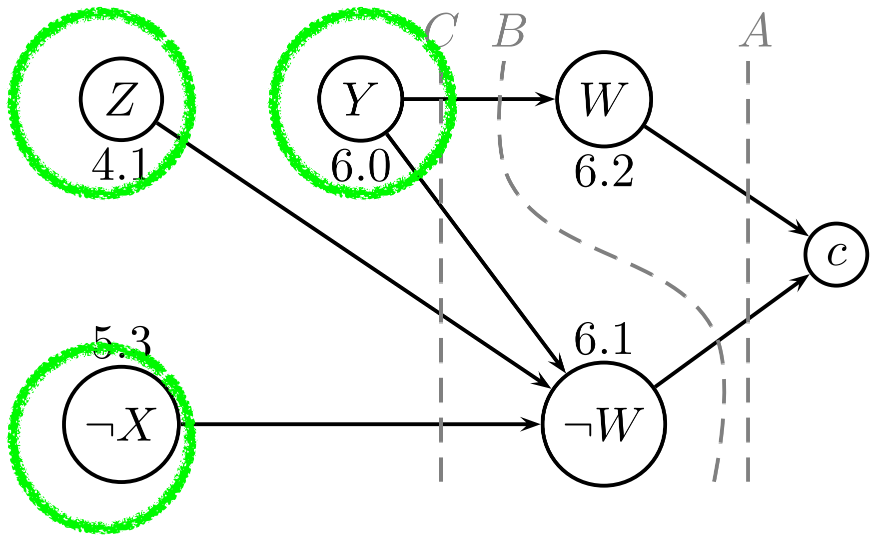
First UIP Cut

- Now we have learnt
- X ∨ ¬Y ∨ ¬Z
- We can add this clause
- If we backtrack to the assignment Y=1
  - the new clause will propagate and set Y=0

- Clauses (fragment)
  - ¬Y ∨ W
  - X ∨ ¬Y ∨ ¬Z ∨ ¬W



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Forgetting

- One problem ....
- If we learn a clause at every failed node
  - and search exponential nodes
  - we end up with exponentially many clauses
- Fortunately...
  - all learnt clauses are *implied*
    - i.e. does not change set of solutions
    - but may help search

Forgetting

- This means we can ...
  - delete any learnt clause ...
  - at any time ...
  - perfectly safely
- So need some kind of forgetting strategy
  - e.g. activity based
  - recently propagated clauses less likely to be forgotten

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VSIDS

- VSIDS means...
  - Variable State Independent, Decaying Sum
- Most common example of an
  - activity based heuristic
  - heuristic for choice of branching literal
- Idea is to choose the most "active" variables in some sense

VSIDS **WARNING**

- VSIDS comes from Chaff
  - and like watched literals is included in the Chaff patent
  - IANAL (I am not a lawyer)
  - So don't believe anything I tell you about the legal position

VSIDS

- Activity based heuristic
- Give each literal a counter.
  - Set all counters to 0
- For each new learnt clause
  - Increment counter for each literal in clause
- When we need a search decision
  - choose literal with highest counter
- Every once in a while ...
  - reduce all counters by a constant factor
  - so that inactive literals decay over time

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