Integrating CP and Mathematical Programming

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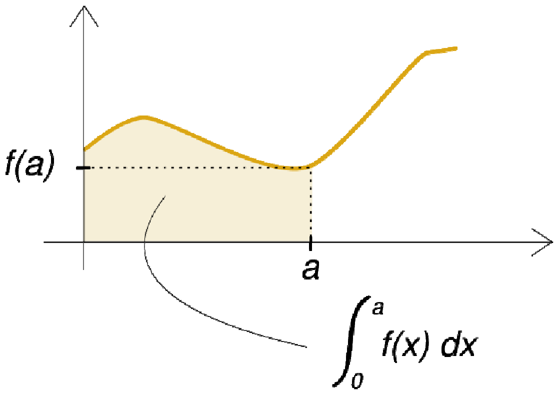


Why Integrate CP and MP?

Complementary strengths

Computational advantages

Outline of the Tutorial

Complementary Strengths

- CP:
  - Inference methods
  - Modeling
  - Exploits local structure
- MP:
  - Relaxation methods
  - Tools for filtering
  - Duality theory
  - Exploits global structure

Let's bring them together!

Computational Advantage of Integrating CP and MP

Using CP + relaxation from MP

Problem	*Relaxation*	*Speedup*
Lesson timetabling	Assignment + reduced cost variable fixing	2 to 50 times faster than CP
Production planning with piecewise linear costs	Convex hull	20 to 120 times faster than MILP (CPLEX 12). Search tree 1000-8000 times smaller
Automatic digital recording	Lagrangean	1 to 10 times faster than MILP, which is faster than CP.

Computational Advantage of Integrating CP and MP

Using CP + relaxation from MP

Problem	*Relaxation*	*Speedup*
Radiation therapy	Lagrangean	10 times faster than CP, MILP
Stable set	Semidefinite programming	Better than CP in less time
Structural design (nonlinear & discrete)	Linear quasi-relaxation + logic cuts	Up to 600 times faster than MILP, GO software 2 problems: <6 min vs >20 hrs for MILP

Computational Advantage of Integrating CP and MP

Using CP-based Branch and Price

Problem	*Speedup*
Urban transit crew scheduling	Optimal schedule for twice as many trips as traditional branch and price
Traveling tournament problem	First to solve 8-team instance

Computational Advantage of Integrating CP and MP

Using Benders methods

Problem	*Method*	*Speedup*
Min-cost machine assignment & scheduling	MILP/CP Benders	20 to 1000 times faster than CP, MILP
Same	SIMPL implementation	Solved some problems in < 1 sec that are intractable for CP, MILP
Polypropylene batch scheduling at BASF	MILP/CP Benders	Solved previously insoluble problem in 10 min

Computational Advantage of Integrating CP and MP

Using Benders methods

Problem	*Method*	*Speedup*
Single-machine scheduling	MILP/CP Benders	Solved much longer time horizons than MILP, CP
Facility assignment and resource-constrained scheduling (min cost, min makespan)	MILP/CP Benders + subproblem relaxations	100-1000 times faster than CP, MILP
Sports scheduling	MILP/CP Benders	Several orders of magnitude relative to state of the art

Software for Integrating CP and MP

- ECLiPSe
  - Exchanges information between ECLiPSEe solver, Xpress-MP
- OPL Studio (IBM)
  - Combines CPLEX and ILOG CP Optimizer with script language
- Xpress-Mosel (FICO)
  - Combines Xpress-MP, Xpress-Kalis with low-level modeling
- G12 (NICTA)
  - Maps problem into script for cooperating solvers
- SIMPL (CMU)
  - Full integration with high-level modeling (prototype)
- SCIP (ZIB)
  - Combines MILP and CP-based propagation

Outline of the Tutorial

- Why Integrate OR and CP?
- A Glimpse at CP
- Initial Example: Integrated Methods
- CP Concepts
- CP Filtering Algorithms
- Linear Relaxation and CP
- Mixed Integer/Linear Modeling
- Network Flows and Filtering
- Integral Polyhedra
- Cutting Planes
- Lagrangean Relaxation and CP
- Dynamic Programming in CP
- CP-based Branch and Price
- CP-based Benders Decomposition

Detailed Outline

- Why Integrate OR and CP?
  - Complementary strengths
  - Computational advantages
  - Outline of the tutorial
- A Glimpse at CP
  - Early successes
  - Advantages and disadvantages
- Initial Example: Integrated Methods
  - Freight Transfer
  - Bounds Propagation
  - Cutting Planes
  - Branch-infer-and-relax Tree

Detailed Outline

- CP Concepts
  - Consistency
  - Hyperarc Consistency
  - Modeling Examples
- CP Filtering Algorithms
  - Element
  - Alldiff
  - Disjunctive Scheduling
  - Cumulative Scheduling
- Linear Relaxation and CP
  - Why relax?
  - Algebraic Analysis of LP
  - Linear Programming Duality
  - LP-Based Domain Filtering
  - Example: Single-Vehicle Routing
  - Disjunctions of Linear Systems

Detailed Outline

- Mixed Integer/Linear Modeling
  - MILP Representability
  - 4.2 Disjunctive Modeling
  - 4.3 Knapsack Modeling
- Network Flows and Filtering
  - Min Cost Network Flow
  - Max Flow
  - Filtering: Cardinality
  - Filtering: Sequence
- Integral Polyhedra
  - Total Unimodularity
  - Network Flow Matrices
  - Interval Matrices

Detailed Outline

- Cutting Planes
  - 0-1 Knapsack Cuts
  - Gomory Cuts
  - Mixed Integer Rounding Cuts
  - Example: Product Configuration
- Lagrangean Relaxation and CP
  - Lagrangean Duality
  - Properties of the Lagrangean Dual
  - Example: Fast Linear Programming
  - Domain Filtering
  - Example: Continuous Global Optimization

Detailed Outline

- Dynamic Programming in CP
  - Example: Capital Budgeting
  - Domain Filtering
  - Recursive Optimization
  - Filtering: Stretch
  - Filtering: Regular
- CP-based Branch and Price
  - Basic Idea
  - Example: Airline Crew Scheduling
- CP-based Benders Decomposition
  - Benders Decomposition in the Abstract
  - Classical Benders Decomposition
  - Example: Machine Scheduling

Background Reading

This tutorial is based on:

- J. N. Hooker, *Integrated Methods for Optimization*, 2nd ed., Springer (to appear 2011). Contains exercises.

A Glimpse at Constraint Programming

Early Successes

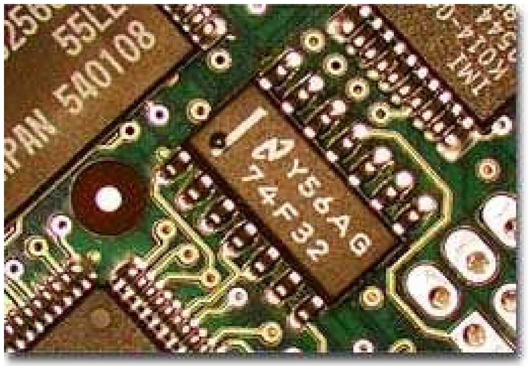
Advantages and Disadvantages

What is constraint programming?

- It is a relatively new technology developed in the computer science and artificial intelligence communities.
- It has found an important role in scheduling, logistics and supply chain management.

Early commercial successes

- Circuit design (Siemens)



- Container port scheduling (Hong Kong and Singapore)

- Real-time control (Siemens, Xerox)

Applications

- Job shop scheduling
- Assembly line smoothing and balancing
- Cellular frequency assignment
- Nurse scheduling
- Shift planning
- Maintenance planning

- Airline crew rostering and scheduling
- Airport gate allocation and stand planning

Applications

- Production scheduling
  - chemicals
  - aviation
  - oil refining
  - steel
  - lumber
  - photographic plates
  - tires
- Transport scheduling (food, nuclear fuel)
- Warehouse management
- Course timetabling

Advantages and Disadvantages

CP vs. Mathematical Programming

MP	CP
Numerical calculation	Logic processing
Relaxation	Inference (filtering, constraint propagation)
Atomistic modeling (linear inequalities)	High-level modeling (global constraints)
Branching	Branching
Independence of model and algorithm	Constraint-based processing

Programming $\neq$ programming

- In **constraint programming**:
  - *programming* = a form of computer programming (constraint-based processing)
- In **mathematical programming**:
  - *programming* = logistics planning (historically)

CP vs. MP

- In **mathematical programming**, equations (constraints) describe the problem but don't tell how to solve it.

- In **constraint programming**, each constraint invokes a procedure that screens out unacceptable solutions.
  - Much as each line of a computer program invokes an operation.

Advantages of CP

- Better at sequencing and scheduling
  - …where MP methods have weak relaxations.
- Adding messy constraints makes the problem easier.
  - The more constraints, the better.
- More powerful modeling language.
  - Global constraints lead to succinct models.
  - Constraints convey problem structure to the solver.
- “Better at highly-constrained problems”
  - Misleading – better when constraints propagate well, or when constraints have few variables.

Disdvantages of CP

- Weaker for continuous variables.
    - Due to lack of numerical techniques
- May fail when constraints contain many variables.
    - These constraints don't propagate well.
- Often not good for finding optimal solutions.
    - Due to lack of relaxation technology.
- May not scale up
    - Discrete combinatorial methods
- Software is not robust
    - Younger field

Obvious solution…

- Integrate CP and MP.

Trends

- CP is better known in continental Europe, Asia.
    - Less known in North America, seen as threat to OR.
- CP/MP integration is growing
    - Eclipse, Mozart, OPL Studio, SIMPL, SCIP, BARON
- Heuristic methods increasingly important in CP
    - Discrete combinatorial methods
- MP/CP/heuristics may become a single technology.

Initial Example: Integrated Methods

Freight Transfer

Bounds Propagation

Cutting Planes

Branch-infer-and-relax Tree

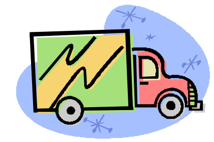
Example: Freight Transfer

- Transport 42 tons of freight overnight in trucks that come in 4 sizes…

Truck size	Number available	Capacity (tons)	Cost per truck
1	3	7	90
2	3	5	60
3	3	4	50
4	3	3	40

- 8 loading docks available.
- Allocate 3 loading docks to the largest trucks even if only 1 or 2 of these trucks are used.

Number of trucks of type 1



$$\begin{aligned}
&\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4 \\
&7x_1 + 5x_2 + 4x_3 + 3x_4 \ge 42 \\
&x_1 + x_2 + x_3 + x_4 \le 8 \\
&(1 \le x_1 \le 2) \Rightarrow (x_2 + x_3 + x_4 \le 5) \\
&x_i \in \{0,1,2,3\}
\end{aligned}$$

Knapsack covering constraint

Knapsack packing constraint

Conditional constraint

Truck type	Number available	Capacity (tons)	Cost per truck
1	3	7	90
2	3	5	60
3	3	4	50
4	3	3	40

Bounds propagation

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$

$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$

$$x_1 + x_2 + x_3 + x_4 \leq 8$$

$$(1 \leq x_1 \leq 2) \Rightarrow (x_2 + x_3 + x_4 \leq 5)$$

$$x_i \in \{0,1,2,3\}$$

$$x_1 \geq \left\lceil \frac{42 - 5\cdot 3 - 4\cdot 3 - 3\cdot 3}{7} \right\rceil = 1$$

Bounds propagation

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$

$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$

$$x_1 + x_2 + x_3 + x_4 \leq 8$$

$$(1 \leq x_1 \leq 2) \Rightarrow (x_2 + x_3 + x_4 \leq 5)$$

$$x_1 \in \{1,2,3\}, \quad x_2, x_3, x_4 \in \{0,1,2,3\}$$

Reduced domain

$$x_1 \geq \left\lceil \frac{42 - 5\cdot 3 - 4\cdot 3 - 3\cdot 3}{7} \right\rceil = 1$$

Bounds consistency

- Let $\{L_j, \ldots, U_j\}$ be the domain of x_j
- A constraint set is **bounds consistent** if for each j :
  - $x_j = L_j$ in some feasible solution and
  - $x_j = U_j$ in some feasible solution.
- Bounds consistency $\Rightarrow$ we will not set x_j to any infeasible values during branching.
- Bounds propagation achieves bounds consistency for a **single inequality**.
  - $7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$ is bounds consistent when the domains are $x_1 \in \{1,2,3\}$ and $x_2, x_3, x_4 \in \{0,1,2,3\}$.
- But not necessarily for a **set** of inequalities.

Bounds consistency

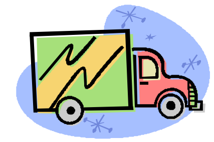
- Bounds propagation may not achieve bounds consistency for a set of constraints.
- Consider set of inequalities

$$x_1 + x_2 \geq 1$$
$$x_1 - x_2 \geq 0$$

with domains x_1, $x_2 \in \{0,1\}$, solutions $(x_1,x_2) = (1,0), (1,1)$.

- Bounds propagation has no effect on the domains.
- But constraint set is not bounds consistent because $x_1 = 0$ in no feasible solution.

Cutting Planes



Begin with continuous relaxation

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$0 \leq x_i \leq 3, \quad x_1 \geq 1$$

Replace domains with bounds

This is a linear programming problem, which is easy to solve.

Its optimal value provides a lower bound on optimal value of original problem.

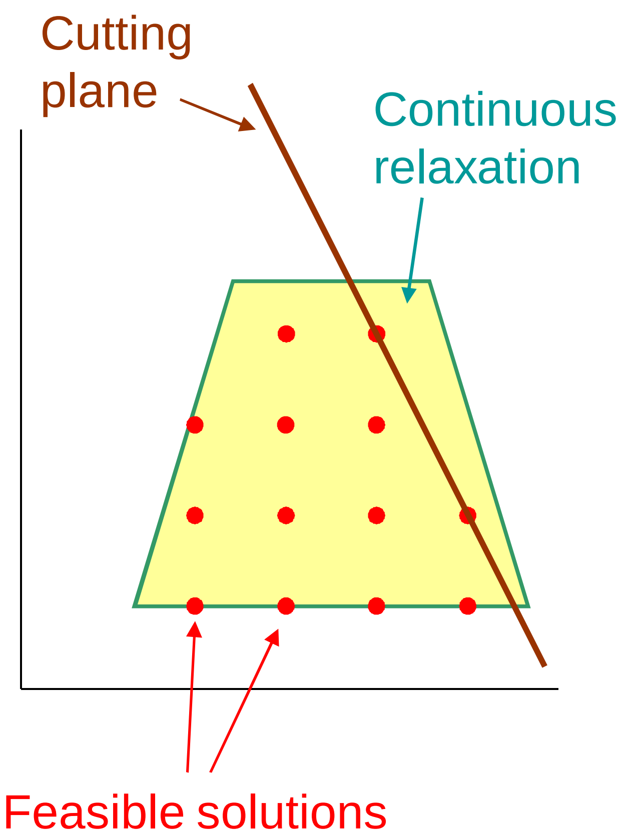
Cutting planes (valid inequalities)

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$0 \leq x_i \leq 3, \quad x_1 \geq 1$$

We can create a **tighter** relaxation (larger minimum value) with the addition of **cutting planes**.

Cutting planes (valid inequalities)

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$0 \leq x_i \leq 3, \quad x_1 \geq 1$$



All feasible solutions of the original problem satisfy a cutting plane (i.e., it is **valid**).

But a cutting plane may exclude ("**cut off**") solutions of the continuous relaxation.

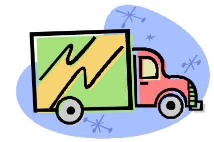
Cutting planes (valid inequalities)

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$0 \leq x_i \leq 3, \quad x_1 \geq 1$$

{1,2} is a **packing**

…because $7x_1 + 5x_2$ alone cannot satisfy the inequality, even with $x_1 = x_2 = 3$.

Cutting planes (valid inequalities)



$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$0 \leq x_i \leq 3, \quad x_1 \geq 1$$

{1,2} is a **packing**

So, $4x_3 + 3x_4 \geq 42 - (7 \cdot 3 + 5 \cdot 3)$

Knapsack cut

which implies

$$x_3 + x_4 \geq \left\lceil \frac{42 - (7 \cdot 3 + 5 \cdot 3)}{\max\{4,3\}} \right\rceil = 2$$

Cutting planes (valid inequalities)

Let x_i have domain $[L_i, U_i]$ and let $a \geq 0$.

In general, a **packing** P for $ax \geq a_0$ satisfies

$$\sum_{i \notin P} a_i x_i \geq a_0 - \sum_{i \in P} a_i U_i$$

and generates a **general integer knapsack cut**

$$\sum_{i \notin P} x_i \geq \left\lceil \frac{a_0 - \sum_{i \in P} a_i U_i}{\max_{i \notin P} \{a_i\}} \right\rceil$$

Cutting planes (valid inequalities)

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$0 \leq x_i \leq 3, \quad x_1 \geq 1$$

Maximal Packings	**Knapsack cuts**
{1,2}	$x_3 + x_4 \geq 2$
{1,3}	$x_2 + x_4 \geq 2$
{1,4}	$x_2 + x_3 \geq 3$
{2,3,4}	$x_1 \geq 1$

Propagated bound

Knapsack cuts corresponding to nonmaximal packings can be nonredundant

Continuous relaxation with cuts

$$\min\ 90x_1 + 60x_2 + 50x_3 + 40x_4$$
$$7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 42$$
$$x_1 + x_2 + x_3 + x_4 \leq 8$$
$$0 \leq x_i \leq 3, \quad x_1 \geq 1$$
$$\boxed{\begin{array}{l} x_3 + x_4 \geq 2 \\ x_2 + x_4 \geq 2 \\ x_2 + x_3 \geq 3 \end{array}}$$

Knapsack cuts

Optimal value of 523.3 is a lower bound on optimal value of original problem.

Branch-infer-and-relax tree

Propagate bounds and solve relaxation of original problem.

$x_1 \in \{\ \ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2⅓,3,2⅔,0)$
value = 523⅓

Branch-infer-and-relax tree

Branch on a variable with nonintegral value in the relaxation.

$x_1 \in \{\ \ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2⅓,3,2⅔,0)$
value = 523⅓

$x_1 \in \{1,2\}$ $x_1 = 3$

Branch-infer-and-relax tree

$x_1 \in \{\ \ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2\frac{1}{3},3,2\frac{2}{3},0)$
value = $523\frac{1}{3}$

$x_1 \in \{1,2\}$

$x_1 = 3$

$x_1 \in \{\ \}$
$x_2 \in \{\ \}$
$x_3 \in \{\ \}$
$x_4 \in \{\ \}$
infeasible

Propagate bounds and solve relaxation.

Since relaxation is infeasible, backtrack.

Branch-infer-and-relax tree

Propagate bounds and solve relaxation.

Branch on nonintegral variable.

$x_1 \in \{\ \ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2⅓,3,2⅔,0)$
value = 523⅓

$x_1 \in \{1,2\}$

$x_1 \in \{\ \}$
$x_2 \in \{\ \}$
$x_3 \in \{\ \}$
$x_4 \in \{\ \}$
infeasible

$x_1 = 3$

$x_1 \in \{\ \ \ \ 3\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (3,2.6,2,0)$
value = 526

$x_2 \in \{0,1,2\}$

$x_2 = 3$

Branch-infer-and-relax tree

Branch again.

$x_1 \in \{\ \ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2\frac{1}{3},3,2\frac{2}{3},0)$
value = $523\frac{1}{3}$

$x_1 \in \{1,2\}$

$x_1 \in \{\ \}$
$x_2 \in \{\ \}$
$x_3 \in \{\ \}$
$x_4 \in \{\ \}$
infeasible

$x_1 = 3$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (3,2.6,2,0)$
value = 526

$x_2 \in \{0,1,2\}$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{012\ \}$
$x_3 \in \{\ 123\}$
$x_4 \in \{0123\}$
$x = (3,2,2\frac{3}{4},0)$
value = $527\frac{1}{2}$

$x_3 \in \{1,2\}$

$x_3 = 3$

$x_2 = 3$

Branch-infer-and-relax tree

Solution of relaxation is integral and therefore feasible in the original problem.

This becomes the **incumbent solution**.

$x_1 \in \{\ \ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2⅓,3,2⅔,0)$
value = 523⅓

$x_1 \in \{1,2\}$

$x_1 \in \{\ \}$
$x_2 \in \{\ \}$
$x_3 \in \{\ \}$
$x_4 \in \{\ \}$
infeasible

$x_1 = 3$

$x_1 \in \{\ \ \ \ 3\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (3,2.6,2,0)$
value = 526

$x_2 \in \{0,1,2\}$

$x_1 \in \{\ \ \ \ 3\}$
$x_2 \in \{012\ \}$
$x_3 \in \{\ \ 123\}$
$x_4 \in \{0123\}$
$x = (3,2,2¾,0)$
value = 527½

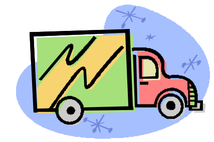
$x_2 = 3$

$x_3 \in \{1,2\}$

$x_1 \in \{\ \ \ \ 3\}$
$x_2 \in \{\ 12\ \}$
$x_3 \in \{\ 12\ \}$
$x_4 \in \{\ 123\}$
$x = (3,2,2,1)$
value = 530
feasible solution

$x_3 = 3$

Branch-infer-and-relax tree



Solution is nonintegral, but we can backtrack because value of relaxation is no better than incumbent solution.

$x_1 \in \{\ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2\frac{1}{3}, 3, 2\frac{2}{3}, 0)$
value = $523\frac{1}{3}$

$x_1 \in \{1,2\}$

$x_1 \in \{\ \}$
$x_2 \in \{\ \}$
$x_3 \in \{\ \}$
$x_4 \in \{\ \}$
infeasible

$x_1 = 3$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (3, 2.6, 2, 0)$
value = 526

$x_2 \in \{0,1,2\}$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{012\ \}$
$x_3 \in \{\ 123\}$
$x_4 \in \{0123\}$
$x = (3, 2, 2\frac{3}{4}, 0)$
value = $527\frac{1}{2}$

$x_2 = 3$

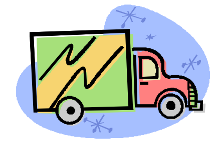
$x_3 \in \{1,2\}$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{\ 12\ \}$
$x_3 \in \{\ 12\ \}$
$x_4 \in \{\ 123\}$
$x = (3, 2, 2, 1)$
value = 530
feasible solution

$x_3 = 3$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{012\ \}$
$x_3 \in \{\ \ \ 3\}$
$x_4 \in \{012\ \}$
$x = (3, 1\frac{1}{2}, 3, \frac{1}{2})$
value = 530
backtrack
due to bound

Branch-infer-and-relax tree



Another feasible solution found.

No better than incumbent solution, which is optimal because search has finished.

$x_1 \in \{\ \ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2⅓,3,2⅔,0)$
value = 523⅓

$x_1 \in \{1,2\}$

$x_1 \in \{\ \}$
$x_2 \in \{\ \}$
$x_3 \in \{\ \}$
$x_4 \in \{\ \}$
infeasible

$x_1 = 3$

$x_1 \in \{\ \ \ \ 3\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (3,2.6,2,0)$
value = 526

$x_2 \in \{0,1,2\}$

$x_1 \in \{\ \ \ \ 3\}$
$x_2 \in \{012\ \}$
$x_3 \in \{\ \ 123\}$
$x_4 \in \{0123\}$
$x = (3,2,2¾,0)$
value = 527½

$x_3 \in \{1,2\}$

$x_1 \in \{\ \ \ \ 3\}$
$x_2 \in \{\ \ 12\ \}$
$x_3 \in \{\ \ 12\ \}$
$x_4 \in \{\ \ 123\}$
$x = (3,2,2,1)$
value = 530
feasible solution

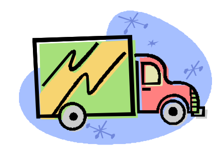
$x_3 = 3$

$x_1 \in \{\ \ \ \ 3\}$
$x_2 \in \{012\ \}$
$x_3 \in \{\ \ \ \ 3\}$
$x_4 \in \{012\ \}$
$x = (3,1½,3,½)$
value = 530
backtrack
due to bound

$x_2 = 3$

$x_1 \in \{\ \ \ \ 3\}$
$x_2 \in \{\ \ \ \ 3\}$
$x_3 \in \{012\ \}$
$x_4 \in \{012\ \}$
$x = (3,3,0,2)$
value = 530
feasible solution

Branch-infer-and-relax tree



Two optimal solutions found.

$x_1 \in \{\ 123\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (2⅓,3,2⅔,0)$
value = 523⅓

$x_1 \in \{1,2\}$

$x_1 \in \{\ \}$
$x_2 \in \{\ \}$
$x_3 \in \{\ \}$
$x_4 \in \{\ \}$
infeasible

$x_1 = 3$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{0123\}$
$x_3 \in \{0123\}$
$x_4 \in \{0123\}$
$x = (3,2.6,2,0)$
value = 526

$x_2 \in \{0,1,2\}$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{012\ \}$
$x_3 \in \{\ 123\}$
$x_4 \in \{0123\}$
$x = (3,2,2¾,0)$
value = 527½

$x_2 = 3$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{\ \ \ 3\}$
$x_3 \in \{012\ \}$
$x_4 \in \{012\ \}$
$x = (3,3,0,2)$
value = 530
feasible solution

$x_3 \in \{1,2\}$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{\ 12\ \}$
$x_3 \in \{\ 12\ \}$
$x_4 \in \{\ 123\}$
$x = (3,2,2,1)$
value = 530
feasible solution

$x_3 = 3$

$x_1 \in \{\ \ \ 3\}$
$x_2 \in \{012\ \}$
$x_3 \in \{\ \ \ 3\}$
$x_4 \in \{012\ \}$
$x = (3,1½,3,½)$
value = 530
backtrack
due to bound

Constraint Programming Concepts

Consistency

Generalized Arc Consistency

Modeling Examples

Consistency

- A constraint set is **consistent** if every partial assignment to the variables that violates no constraint is feasible.
    - i.e., can be extended to a feasible solution.
- Consistency ≠ feasibility
    - Consistency means that any infeasible partial assignment is explicitly ruled out by a constraint.
- Fully consistent constraint sets can be solved **without backtracking**.

Consistency

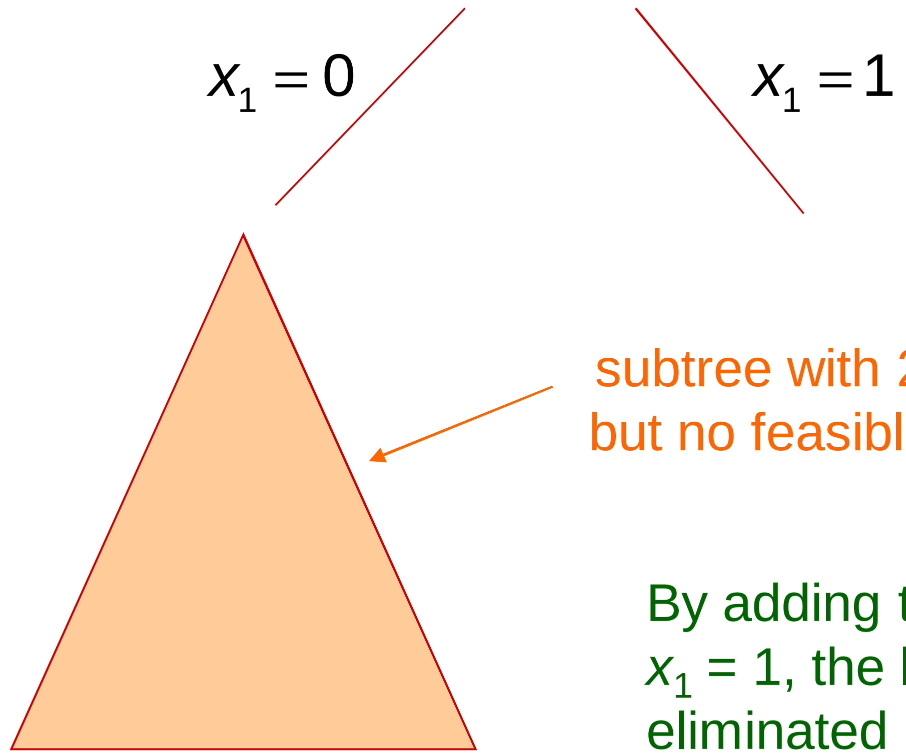
Consider the constraint set

$$x_1 + x_{100} \geq 1$$
$$x_1 - x_{100} \geq 0$$
$$x_j \in \{0,1\}$$

It is not consistent, because $x_1 = 0$ violates no constraint and yet is infeasible (no solution has $x_1 = 0$).

Adding the constraint $x_1 = 1$ makes the set consistent.

$$x_1 + x_{100} \geq 1$$
$$x_1 - x_{100} \geq 1$$
other constraints
$$x_j \in \{0,1\}$$



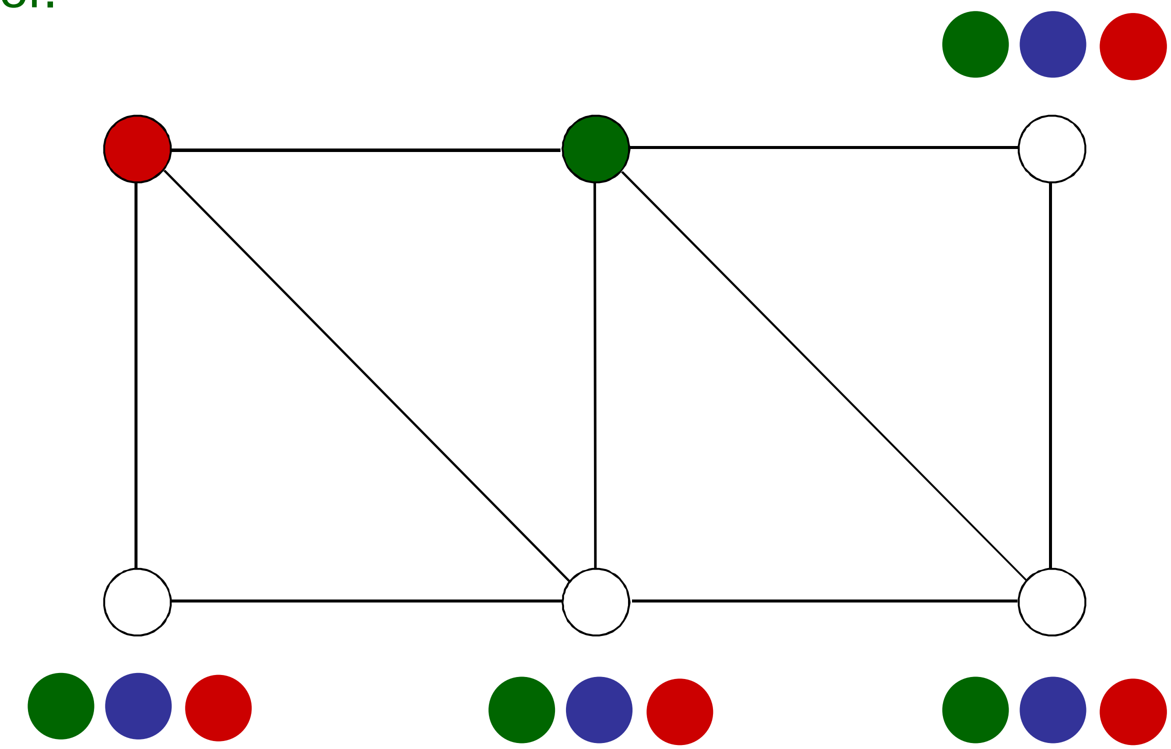
subtree with 2^{99} nodes
but no feasible solution

By adding the constraint
$x_1 = 1$, the left subtree is
eliminated

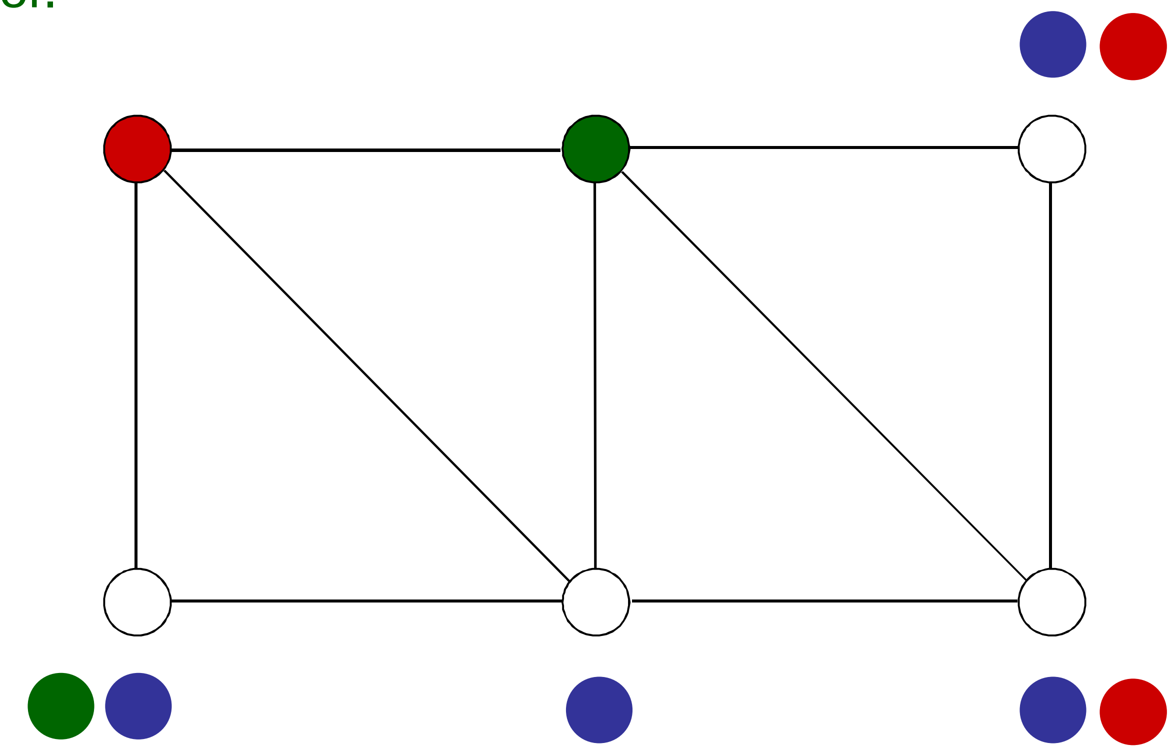
Generalized Arc Consistency (GAC)

- Also known as **hyperarc consistency**.
- A constraint set is **GAC** if every value in every variable domain is part of some feasible solution.
    - That is, the domains are reduced as much as possible.
    - If all constraints are "binary" (contain 2 variables), GAC = arc consistency.
    - Domain reduction is CP's biggest engine.

Graph coloring problem that can be solved by arc consistency maintenance alone. Color nodes with red, green, blue with no two adjacent nodes having the same color.



Graph coloring problem that can be solved by arc consistency maintenance alone. Color nodes with red, green, blue with no two adjacent nodes having the same color.



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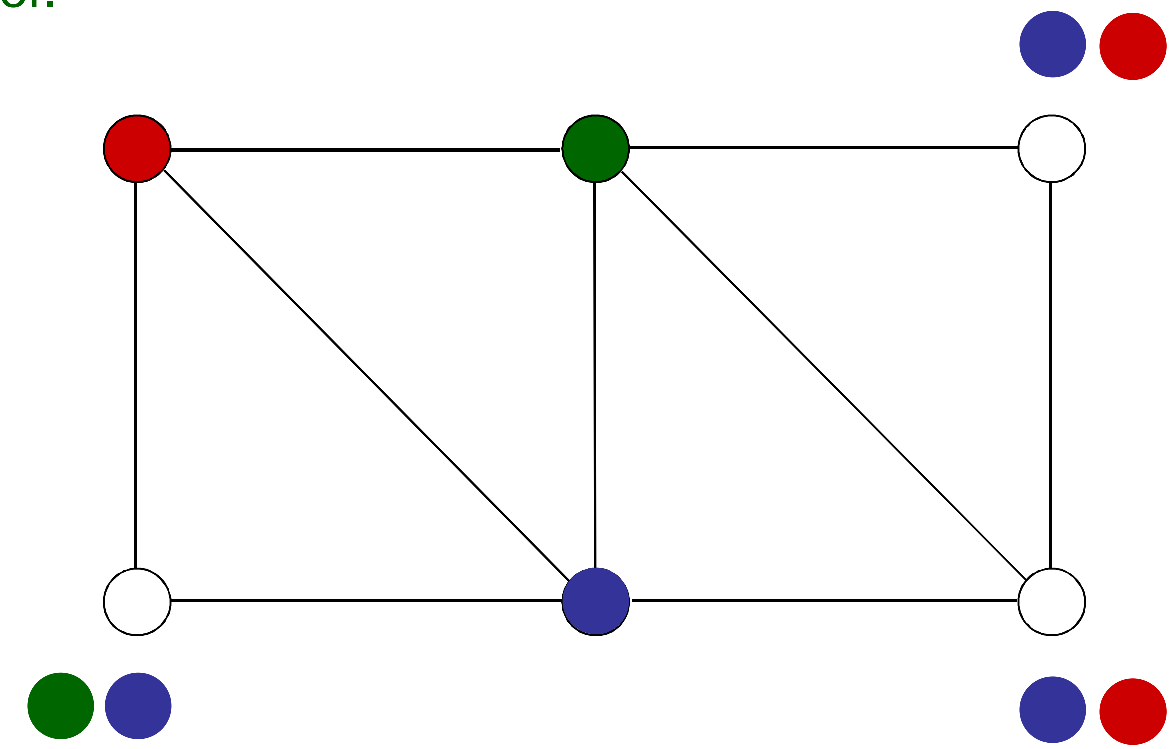
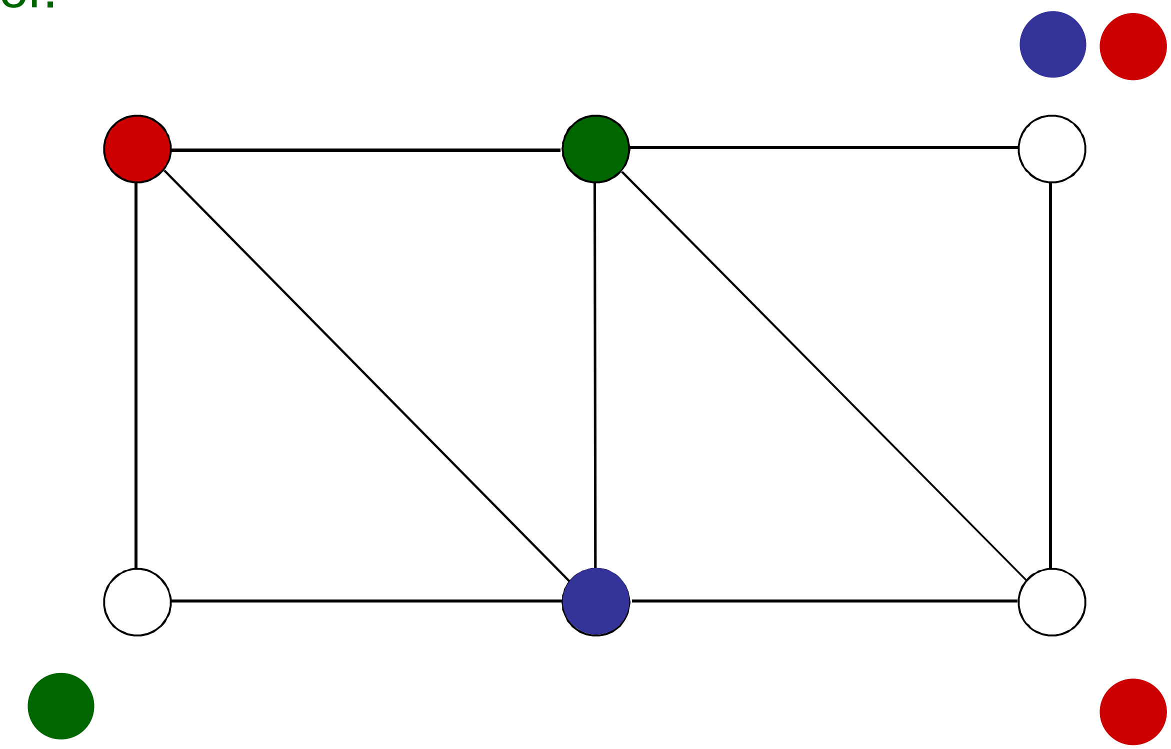
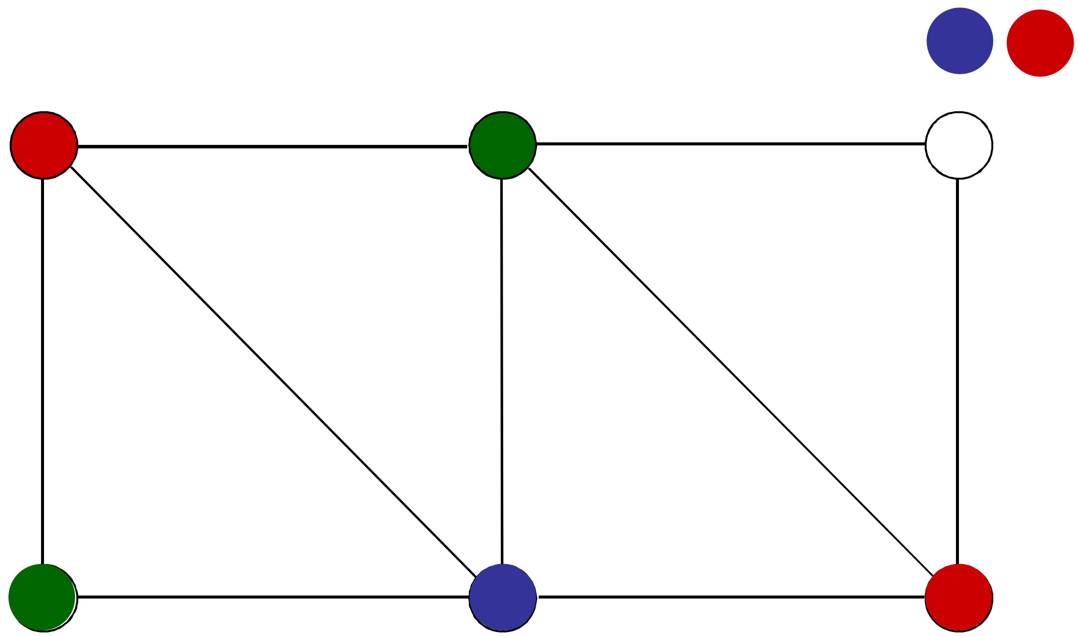


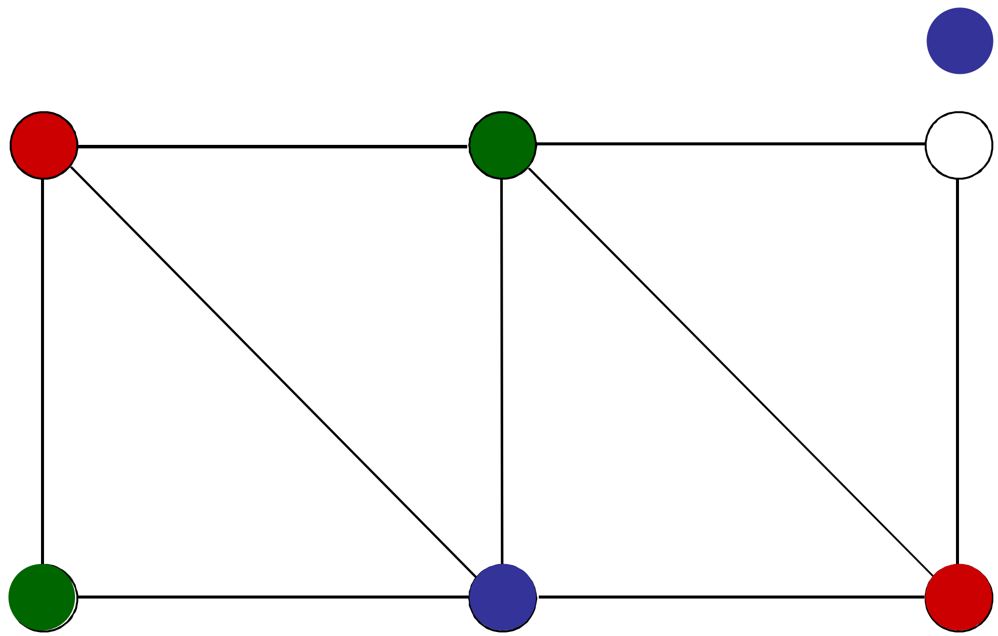
Graph coloring problem that can be solved by arc consistency maintenance alone. Color nodes with red, green, blue with no two adjacent nodes having the same color.



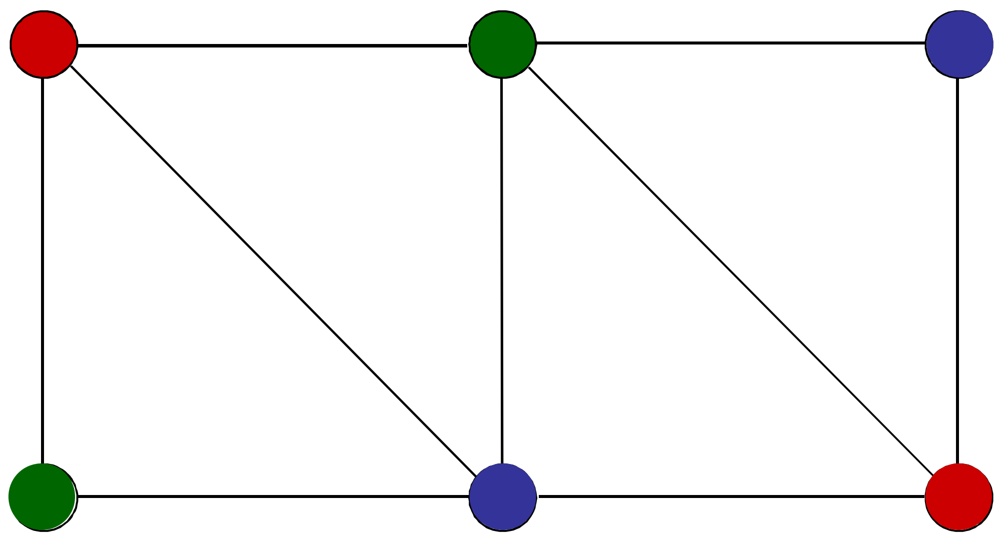
Graph coloring problem that can be solved by arc consistency maintenance alone. Color nodes with red, green, blue with no two adjacent nodes having the same color.



Graph coloring problem that can be solved by arc consistency maintenance alone. Color nodes with red, green, blue with no two adjacent nodes having the same color.



Graph coloring problem that can be solved by arc consistency maintenance alone. Color nodes with red, green, blue with no two adjacent nodes having the same color.



Modeling Examples with Global Constraints

Traveling Salesman

Traveling salesman problem:

Let c_{ij} = distance from city i to city j.

Find the shortest route that visits each of n cities exactly once.

Popular 0-1 model

Let x_{ij} = 1 if city i immediately precedes city j, 0 otherwise

$$
\begin{aligned}
\text{min} \quad & \sum_{ij} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i} x_{ij} = 1, \quad \text{all } j \\
& \sum_{j} x_{ij} = 1, \quad \text{all } i \\
& \sum_{i \in V} \sum_{j \in W} x_{ij} \geq 1, \quad \text{all disjoint } V, W \subset \{1, \ldots, n\} \\
& x_{ij} \in \{0, 1\}
\end{aligned}
$$

Subtour elimination constraints

A CP model

Let y_k = the kth city visited.

The model would be written in a specific constraint programming language but would essentially say:

Variable indices

$$\begin{aligned} \min \quad & \sum_k c_{y_k y_{k+1}} \\ \text{s.t.} \quad & \text{alldiff}(y_1, \ldots, y_n) \\ & y_k \in \{1, \ldots, n\} \end{aligned}$$

"Global" constraint

An alternate CP model

Let y_k = the city visited after city k.

$$\begin{aligned} \min \quad & \sum_k c_{k y_k} \\ \text{s.t.} \quad & \text{circuit}(y_1, \ldots, y_n) \\ & y_k \in \{1, \ldots, n\} \end{aligned}$$

Hamiltonian circuit constraint

Element constraint

The constraint $c_y \leq 5$ can be implemented:

$$z \leq 5$$
$$\text{element}\left(y,(c_1,\ldots,c_n),z\right)$$

Assign z the yth value in the list

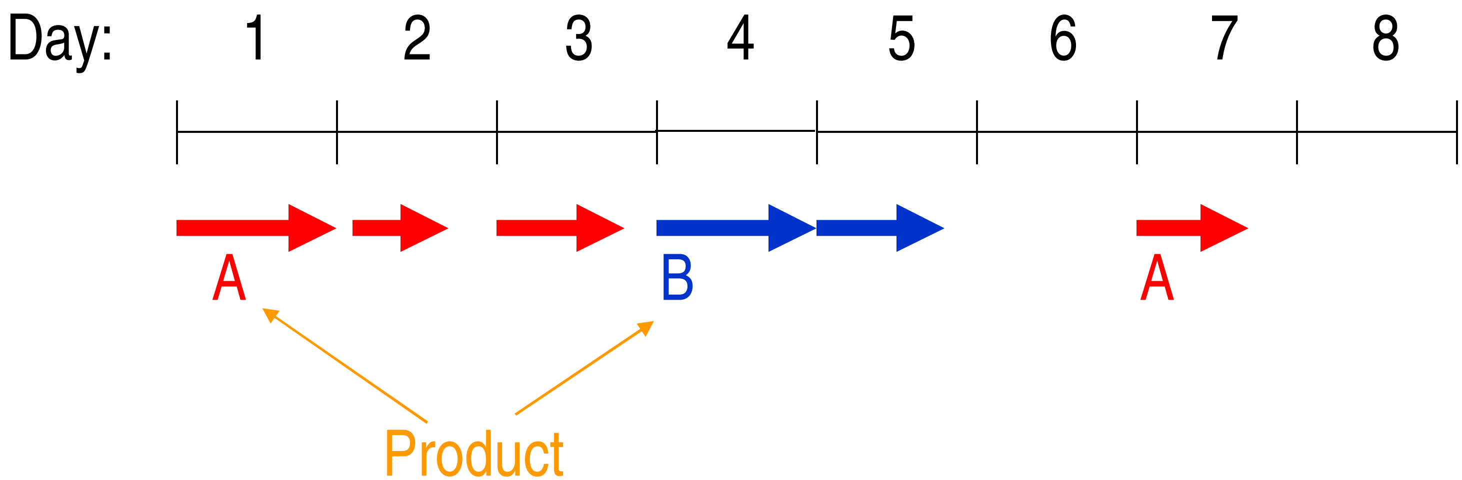
The constraint $x_y \leq 5$ can be implemented

$$z \leq 5$$
$$\text{element}\left(y,(x_1,\ldots,x_n),z\right)$$

Add the constraint $z = x_y$

(this is a slightly different constraint)

Modeling example: Lot sizing and scheduling



- At most one product manufactured on each day.
- Demands for each product on each day.
- Minimize setup + holding cost.

Integer programming model

(Wolsey)

$$\min \sum_{t,i} \left(h_{it} s_{it} + \sum_{j \neq t} q_{ij} \delta_{ijt} \right)$$

Many variables

$$\begin{aligned}
\text{s.t.}\quad & s_{i,t-1} + x_{it} = d_{it} + s_{it}, \quad \text{all } i,t \\
& z_{it} \geq y_{it} - y_{i,t-1}, \quad \text{all } i,t \\
& z_{it} \leq y_{it}, \quad \text{all } i,t \\
& z_{it} \leq 1 - y_{i,t-1}, \quad \text{all } i,t \\
& \delta_{ijt} \geq y_{i,t-1} + y_{jt} - 1, \quad \text{all } i,j,t \\
& \delta_{ijt} \geq y_{i,t-1}, \quad \text{all } i,j,t \\
& \delta_{ijt} \geq y_{jt}, \quad \text{all } i,j,t \\
& x_{it} \leq C y_{it}, \quad \text{all } i,t \\
& \sum_i y_{it} = 1, \quad \text{all } t \\
& y_{it}, z_{it}, \delta_{ijt} \in \{0,1\} \\
& x_{it}, s_{it} \geq 0
\end{aligned}$$

CP model

Minimize holding and setup costs

$$\min \quad \sum_t \left(q_{y_{t-1} y_t} + \sum_i h_i s_{it} \right)$$

Inventory balance

Production capacity

$$\text{s.t.} \quad s_{i,t-1} + x_{it} = d_{it} + s_{it}, \quad \text{all } i,t$$

$$0 \le x_{it} \le C, \quad s_{it} \ge 0, \quad \text{all } i,t$$

$$(y_t \ne i) \rightarrow (x_{it} = 0), \quad \text{all } i,t$$

CP model

Minimize holding and setup costs

Variable indices

$$\min \quad \sum_t \left(q_{y_{t-1} y_t} + \sum_i h_i s_{it} \right)$$

Inventory balance

$$\text{s.t.} \quad s_{i,t-1} + x_{it} = d_{it} + s_{it}, \quad \text{all } i,t$$

Production capacity

$$0 \le x_{it} \le C, \quad s_{it} \ge 0, \quad \text{all } i,t$$

$$(y_t \ne i) \rightarrow (x_{it} = 0), \quad \text{all } i,t$$

Production level of product *i* in period *t*

Product manufactured in period *t*

Cumulative scheduling constraint

- Used for resource-constrained scheduling.
- Total resources consumed by jobs at any one time must not exceed *L*.

$$\text{cumulative}\big((t_1,\ldots,t_n),(p_1,\ldots,p_n),(c_1,\ldots,c_n),L\big)$$

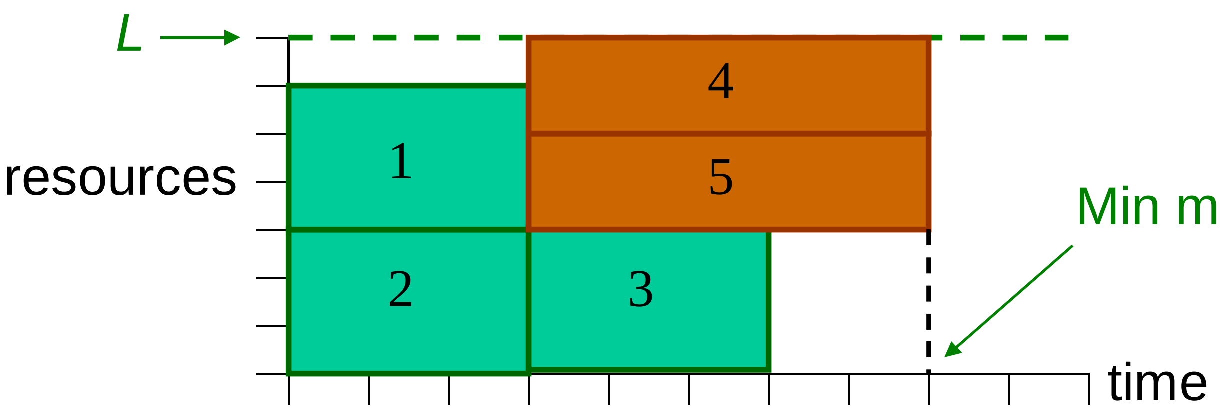
Job start times (variables)

Job processing times

Job resource requirements

Cumulative scheduling constraint

Minimize makespan (no deadlines, all release times = 0):



L

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & \text{cumulative}\big((t_1,\ldots,t_5),(3,3,3,5,5),(3,3,3,2,2),7\big) \\ & z \ge t_1 + 3 \\ & \vdots \\ & z \ge t_5 + 2 \end{aligned}$$

Resources used

Processing times

Job start times

Modeling example: Ship loading

- Will use ILOG's OPL Studio modeling language.
  - Example is from OPL manual.
- The problem
  - Load 34 items on the ship in minimum time (min makespan)
  - Each item requires a certain time and certain number of workers.
  - Total of 8 workers available.

Problem data

Item	Duration	Labor
1	3	4
2	4	4
3	4	3
4	6	4
5	5	5
6	2	5
7	3	4
8	4	3
9	3	4
10	2	8
11	3	4
12	2	5
13	1	4
14	5	3
15	2	3
16	3	3
17	2	6

Item	Duration	Labor
18	2	7
19	1	4
20	1	4
21	1	4
22	2	4
23	4	7
24	5	8
25	2	8
26	1	3
27	1	3
28	2	6
29	1	8
30	3	3
31	2	3
32	1	3
33	2	3
34	2	3

Precedence constraints

1 → 2,4	11 → 13	22 → 23
2 → 3	12 → 13	23 → 24
3 → 5,7	13 → 15,16	24 → 25
4 → 5	14 → 15	25 → 26,30,31,32
5 → 6	15 → 18	26 → 27
6 → 8	16 → 17	27 → 28
7 → 8	17 → 18	28 → 29
8 → 9	18 → 19	30 → 28
9 → 10	18 → 20,21	31 → 28
9 → 14	19 → 23	32 → 33
10 → 11	20 → 23	33 → 34
10 → 12	21 → 22	

Use the cumulative scheduling constraint.

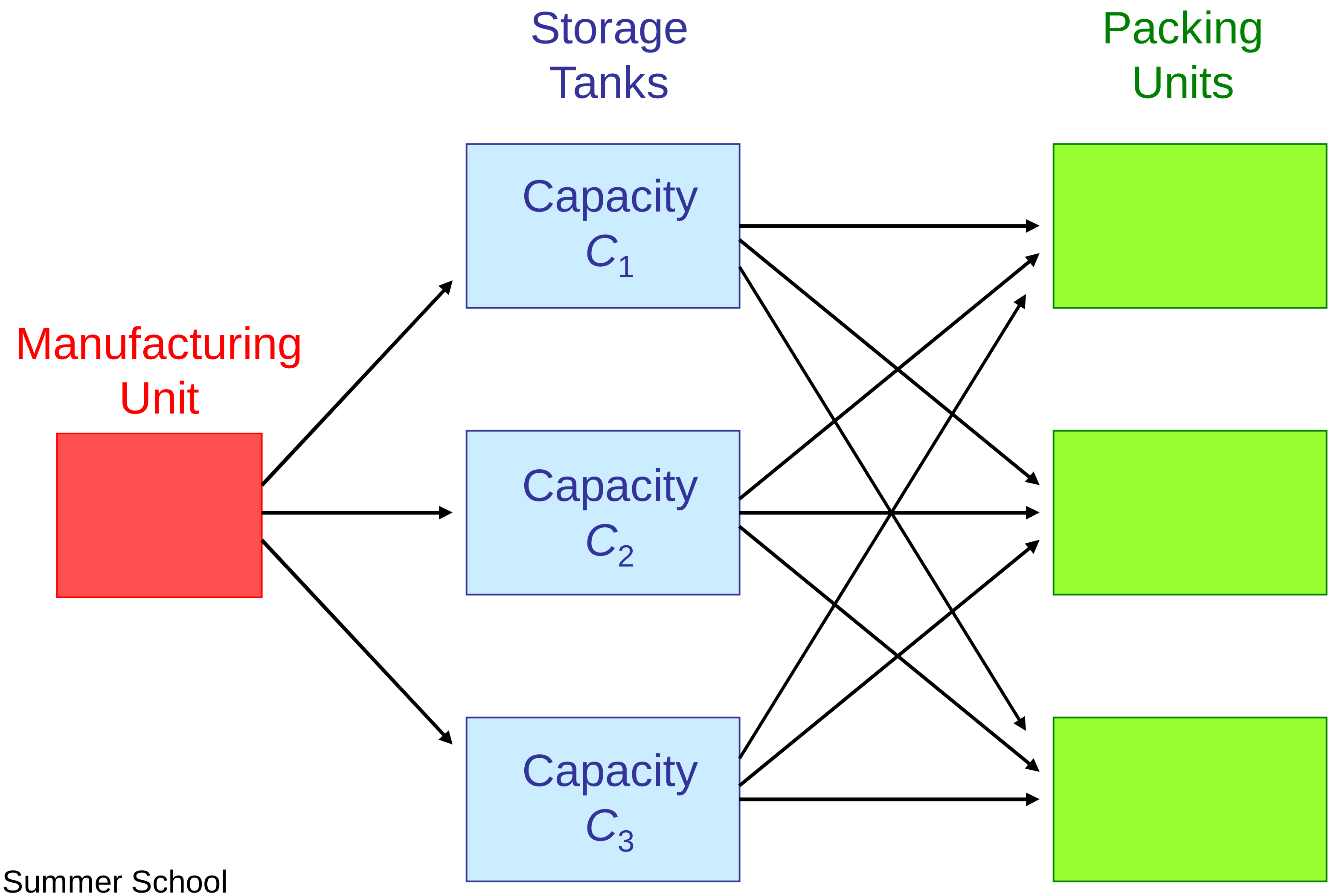
$$
\begin{aligned}
\min \quad & z \\
\text{s.t.} \quad & z \geq t_1 + 3, \quad z \geq t_2 + 4, \ \ \text{etc.} \\
& \text{cumulative}\big((t_1,\ldots,t_{34}),(3,4,\ldots,2),(4,4,\ldots,3),8\big) \\
& t_2 \geq t_1 + 3, \ \ t_4 \geq t_1 + 3, \ \ \text{etc.}
\end{aligned}
$$

OPL model

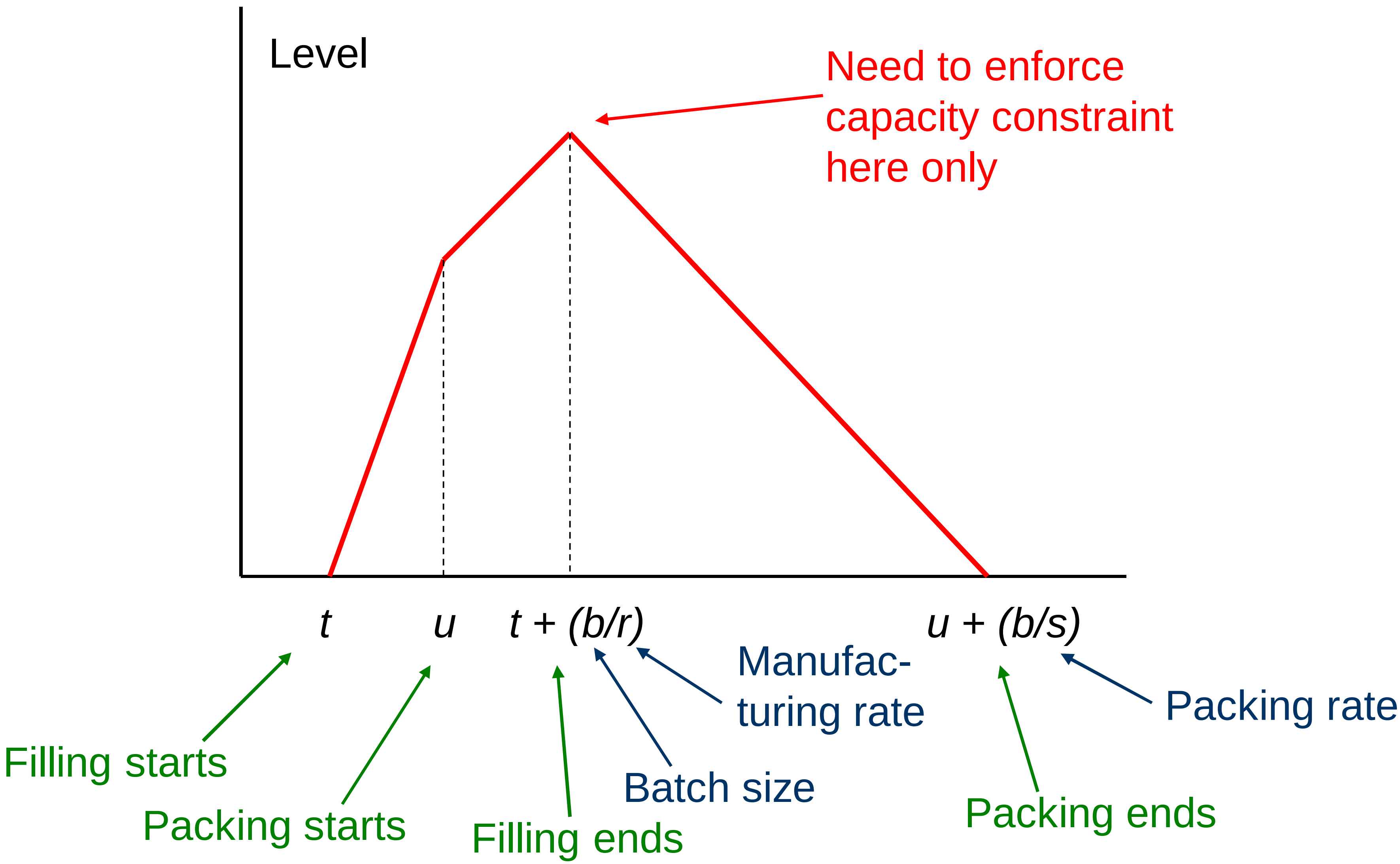
```
int capacity = 8;
int nbTasks = 34;
range Tasks 1..nbTasks;
int duration[Tasks] = [3,4,4,6,…,2];
int totalDuration =
      sum(t in Tasks) duration[t];
int demand[Tasks] = [4,4,3,4,…,3];
struct Precedences {
   int before;
   int after;

}
{Precedences} setOfPrecedences = {
      <1,2>, <1,4>, …, <33,34> };
```

```
scheduleHorizon = totalDuration;
Activity a[t in Tasks](duration[t]);
DiscreteResource res(8);
Activity makespan(0);
minimize
   makespan.end
subject to
   forall(t in Tasks)
      a[t] precedes makespan;
   forall(p in setOfPrecedences)
      a[p.before] precedes a[p.after];
   forall(t in Tasks)
      a[t] requires(demand[t]) res;
};
```

Modeling example: Production scheduling with intermediate storage



Filling of storage tank



$$\min \quad T$$

Makespan

$$\text{s.t.} \quad T \ge u_j + \frac{b_j}{s_j}, \quad \text{all } j$$

$$t_j \ge R_j, \quad \text{all } j$$

Job release time

$$\text{cumulative}(t, v, e, m)$$

m storage tanks

$$v_i = u_i + \frac{b_i}{s_i} - t_i, \quad \text{all } i$$

Job duration

$$b_i\left(1 - \frac{s_i}{r_i}\right) + s_i u_i \le C_i, \quad \text{all } i$$

Tank capacity

$$\text{cumulative}\left(u, \left(\frac{b_1}{s_1}, \ldots, \frac{b_n}{s_n}\right), e, p\right)$$

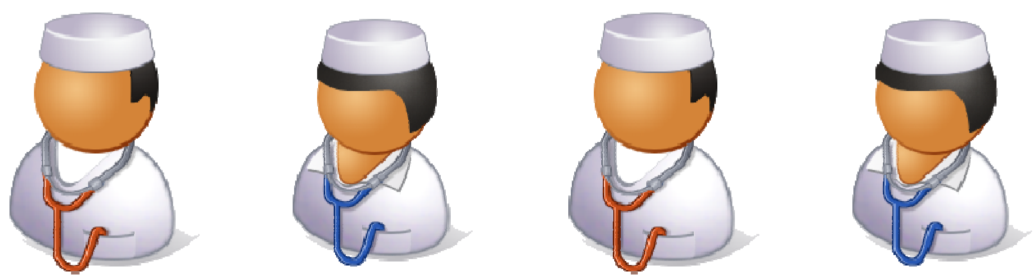
p packing units

$$u_j \ge t_j \ge 0$$

$e = (1,\ldots,1)$

Modeling example: Employee scheduling

- Schedule four nurses in 8-hour shifts.
- A nurse works at most one shift a day, at least 5 days a week.
- Same schedule every week.
- No shift staffed by more than two different nurses in a week.
- A nurse cannot work different shifts on two consecutive days.
- A nurse who works shift 2 or 3 must do so at least two days in a row.



Two ways to view the problem

Assign nurses to shifts

	Sun	Mon	Tue	Wed	Thu	Fri	Sat
Shift 1	A	B	A	A	A	A	A
Shift 2	C	C	C	B	B	B	B
Shift 3	D	D	D	D	C	C	D

Assign shifts to nurses

	Sun	Mon	Tue	Wed	Thu	Fri	Sat
Nurse A	1	0	1	1	1	1	1
Nurse B	0	1	0	2	2	2	2
Nurse C	2	2	2	0	3	3	0
Nurse D	3	3	3	3	0	0	3

0 = day off

Use **both** formulations in the same model!

First, assign nurses to shifts.

Let w_{sd} = nurse assigned to shift s on day d

$$\text{alldiff}(w_{1d}, w_{2d}, w_{3d}), \quad \text{all } d$$

The variables w_{1d}, w_{2d}, w_{3d} take different values

That is, schedule 3 different nurses on each day

Use **both** formulations in the same model!

First, assign nurses to shifts.

Let w_{sd} = nurse assigned to shift s on day d

$$\text{alldiff}(w_{1d}, w_{2d}, w_{3d}), \quad \text{all } d$$
$$\text{cardinality}\big(w \,|\, (A,B,C,D),(5,5,5,5),(6,6,6,6)\big)$$

A occurs at least 5 and at most 6 times in the array *w*, and similarly for B, C, D.

That is, each nurse works at least 5 and at most 6 days a week

Use **both** formulations in the same model!

First, assign nurses to shifts.

Let w_{sd} = nurse assigned to shift s on day d

$$\text{alldiff}\left(w_{1d}, w_{2d}, w_{3d}\right), \quad \text{all } d$$

$$\text{cardinality}\left(w \mid (A,B,C,D),(5,5,5,5),(6,6,6,6)\right)$$

$$\text{nvalues}\left(w_{s,\text{Sun}}, \ldots, w_{s,\text{Sat}} \mid 1,2\right), \quad \text{all } s$$

The variables $w_{s,\text{Sun}}, \ldots, w_{s,\text{Sat}}$ take at least 1 and at most 2 different values.

That is, at least 1 and at most 2 nurses work any given shift.

Remaining constraints are not easily expressed in this notation.

So, assign shifts to nurses.

Let y_{id} = shift assigned to nurse i on day d

$$\text{alldiff}\left(y_{1d}, y_{2d}, y_{3d}\right), \text{ all } d$$

Assign a different nurse to each shift on each day.

This constraint is redundant of previous constraints, but redundant constraints speed solution.

Remaining constraints are not easily expressed in this notation.

So, assign shifts to nurses.

Let y_{id} = shift assigned to nurse i on day d

$$\text{alldiff}\left(y_{1d}, y_{2d}, y_{3d}\right),\ \text{all } d$$
$$\text{stretch}\left(y_{i,\text{Sun}}, \ldots, y_{i,\text{Sat}} \mid (2,3), (2,2), (6,6), P\right),\ \text{all } i$$

Every stretch of 2’s has length between 2 and 6.
Every stretch of 3’s has length between 2 and 6.

So a nurse who works shift 2 or 3 must do so at least two days in a row.

Remaining constraints are not easily expressed in this notation.

So, assign shifts to nurses.

Let y_{id} = shift assigned to nurse i on day d

$$\text{alldiff}\left(y_{1d}, y_{2d}, y_{3d}\right),\ \text{ all } d$$
$$\text{stretch}\left(y_{i,\text{Sun}}, \ldots, y_{i,\text{Sat}} \mid (2,3),(2,2),(6,6),P\right),\ \text{ all } i$$

Here $P = \{(s,0),(0,s) \mid s = 1,2,3\}$

Whenever a stretch of a's immediately precedes a stretch of b's, (a,b) must be one of the pairs in P.

So a nurse cannot switch shifts without taking at least one day off.

Now we must connect the w_{sd} variables to the y_{id} variables.

Use **channeling constraints**:

$$w_{y_{id}d} = i, \quad \text{all } i,d$$

$$y_{w_{sd}d} = s, \quad \text{all } s,d$$

Channeling constraints increase propagation and make the problem easier to solve.

The complete model is:

$$\text{alldiff}\left(w_{1d}, w_{2d}, w_{3d}\right),\ \ \text{all } d$$
$$\text{cardinality}\left(w \mid (A,B,C,D),(5,5,5,5),(6,6,6,6)\right)$$
$$\text{nvalues}\left(w_{s,\text{Sun}},\ldots,w_{s,\text{Sat}} \mid 1,2\right),\ \ \text{all } s$$

$$\text{alldiff}\left(y_{1d}, y_{2d}, y_{3d}\right),\ \ \text{all } d$$
$$\text{stretch}\left(y_{i,\text{Sun}},\ldots,y_{i,\text{Sat}} \mid (2,3),(2,2),(6,6),P\right),\ \ \text{all } i$$

$$w_{y_{id}d} = i,\ \ \text{all } i,d$$
$$y_{w_{sd}d} = s,\ \ \text{all } s,d$$

CP Filtering Algorithms

Element

Alldiff

Disjunctive Scheduling

Cumulative Scheduling

Filtering for element

$$\text{element}\left(y,(x_1,\ldots,x_n),z\right)$$

Variable domains can be easily filtered to maintain GAC.

Domain of z

$$D_z \leftarrow D_z \cap \bigcup_{j \in D_y} D_{x_j}$$

$$D_y \leftarrow D_y \cap \left\{ j \mid D_z \cap D_{x_j} \neq \emptyset \right\}$$

$$D_{x_j} \leftarrow \begin{Bmatrix} D_z & \text{if } D_y = \{j\} \\ D_{x_j} & \text{otherwise} \end{Bmatrix}$$

Filtering for element

Example... $\text{element}\left(y,(x_1,x_2,x_3,x_4),z\right)$

The initial domains are:

$$D_z = \{20,30,60,80,90\}$$
$$D_y = \{1,3,4\}$$
$$D_{x_1} = \{10,50\}$$
$$D_{x_2} = \{10,20\}$$
$$D_{x_3} = \{40,50,80,90\}$$
$$D_{x_4} = \{40,50,70\}$$

The reduced domains are:

$$D_z = \{80,90\}$$
$$D_y = \{3\}$$
$$D_{x_1} = \{10,50\}$$
$$D_{x_2} = \{10,20\}$$
$$D_{x_3} = \{80,90\}$$
$$D_{x_4} = \{40,50,70\}$$

Filtering for alldiff

$$\text{alldiff}\left(y_1,\ldots,y_n\right)$$

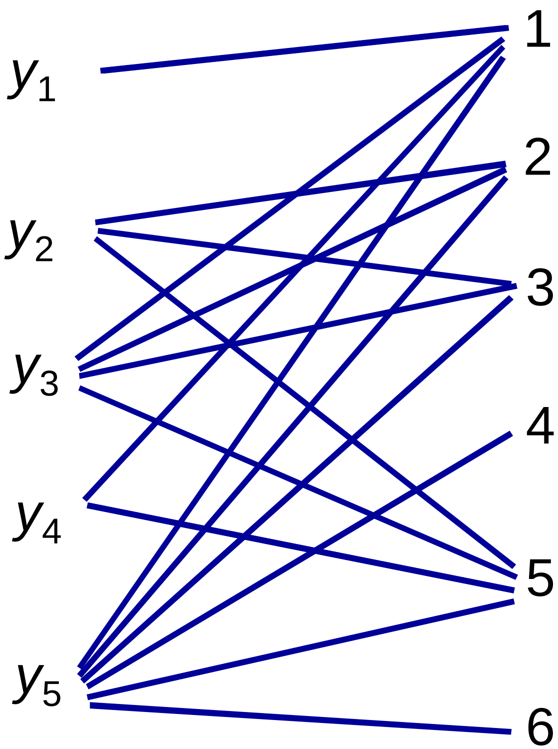
Domains can be filtered with an algorithm based on maximum cardinality bipartite matching and a theorem of Berge.

It is a special case of optimality conditions for max flow.

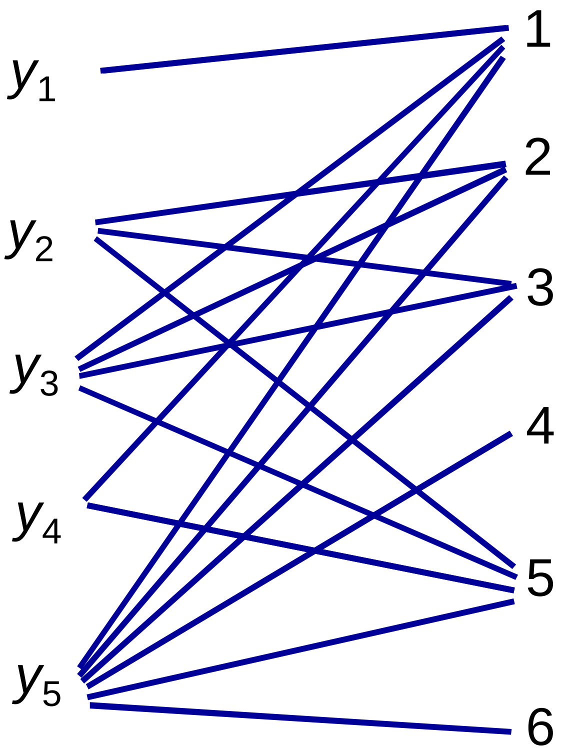
Filtering for alldiff

Consider the domains

$$
\begin{aligned}
y_1 &\in \{1\} \\
y_2 &\in \{2,3,5\} \\
y_3 &\in \{1,2,3,5\} \\
y_4 &\in \{1,5\} \\
y_5 &\in \{1,2,3,4,5,6\}
\end{aligned}
$$

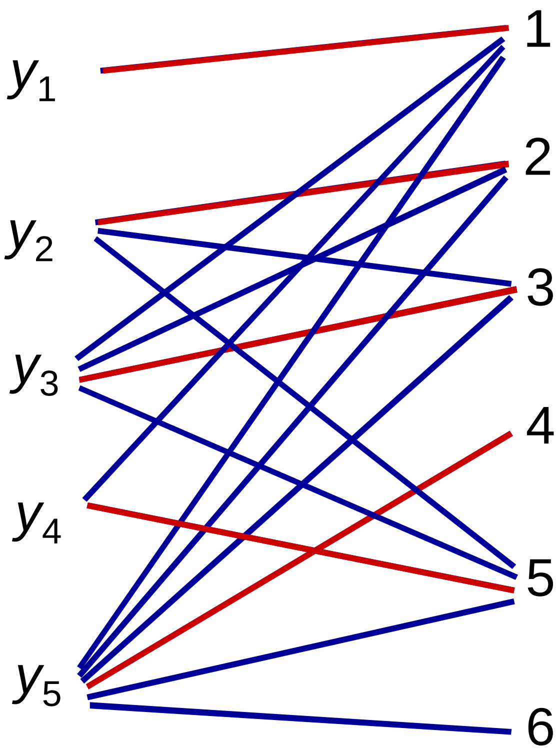


Indicate domains with edges



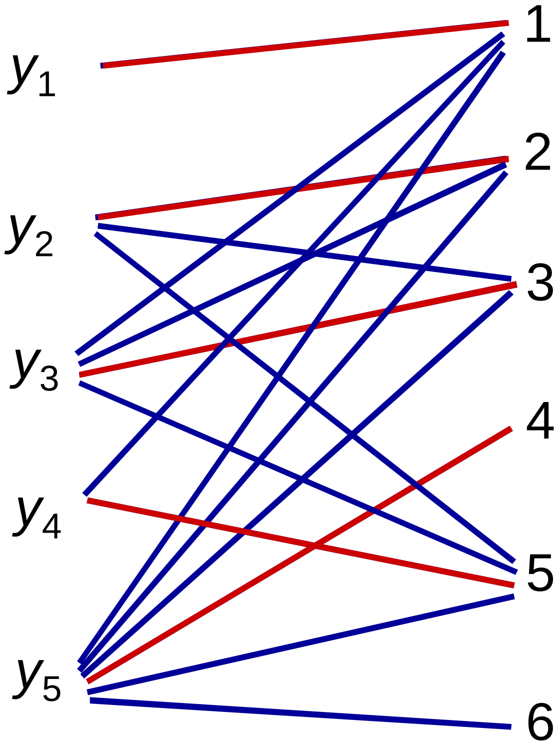
Indicate domains with edges

Find maximum cardinality bipartite matching.



Indicate domains with edges

Find maximum cardinality bipartite matching.



Indicate domains with edges

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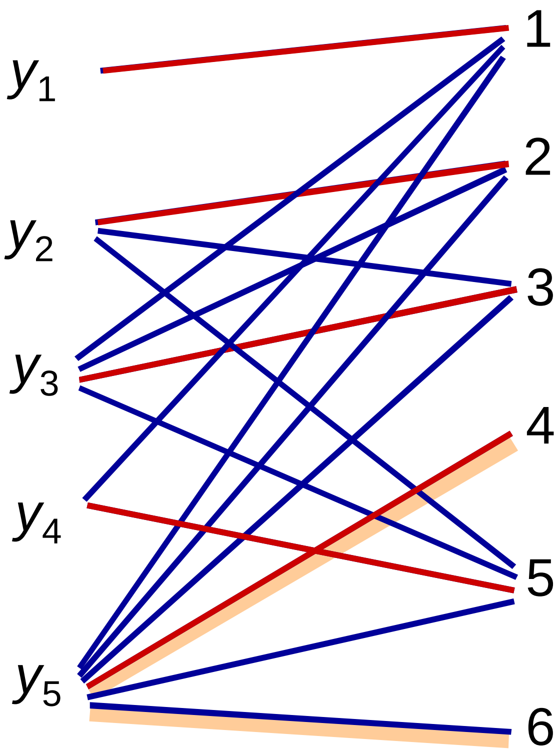
Mark edges in alternating paths that start at an uncovered vertex.



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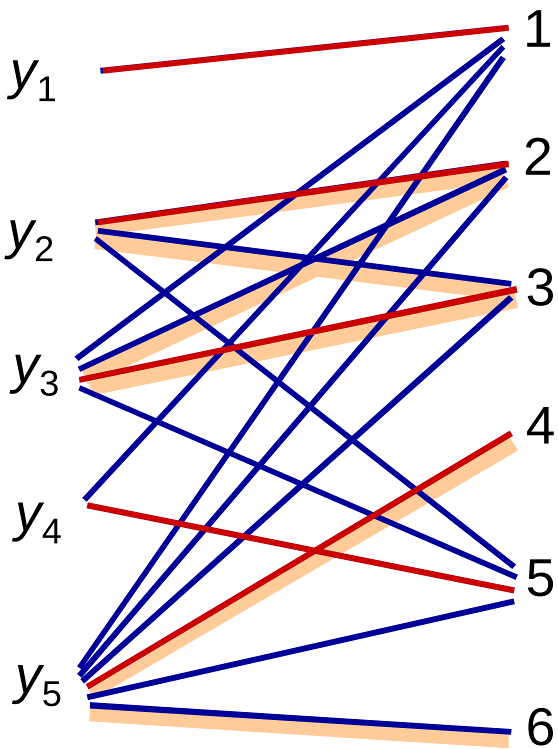

Indicate domains with edges

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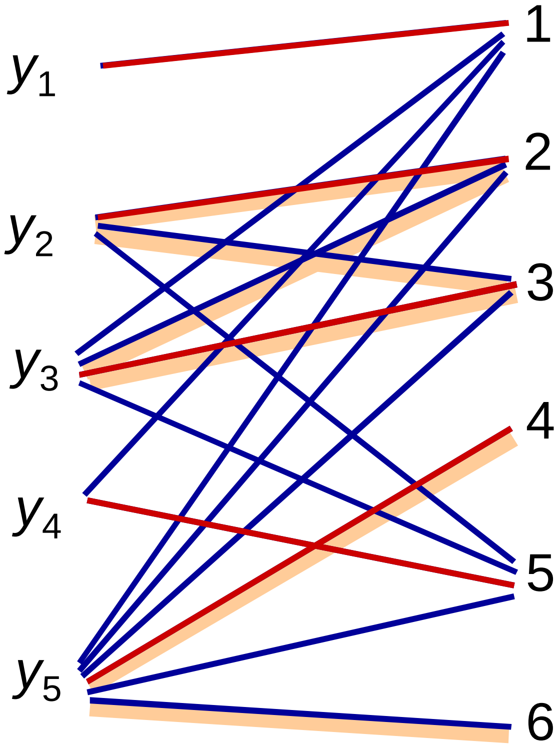
Mark edges in alternating cycles.



Indicate domains with edges

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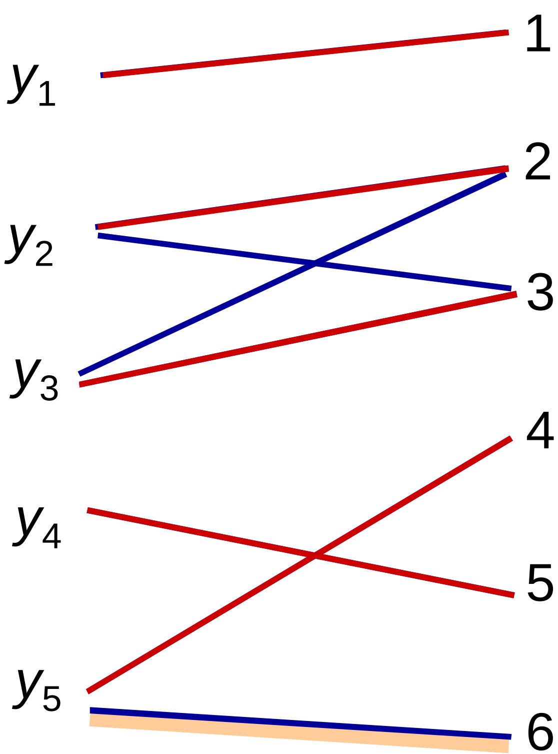
Mark edges in alternating paths that start at an uncovered vertex.

Mark edges in alternating cycles.

Remove unmarked edges not in matching.



Indicate domains with edges

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Mark edges in alternating paths that start at an uncovered vertex.

Mark edges in alternating cycles.

Remove unmarked edges not in matching.

Filtering for alldiff

Domains have been filtered:

$$\begin{array}{ll} y_1 \in \{1\} & \\ y_2 \in \{2,3,5\} & \\ y_3 \in \{1,2,3,5\} & \\ y_4 \in \{1,5\} & \\ y_5 \in \{1,2,3,4,5,6\} & \end{array} \longrightarrow \begin{array}{l} y_1 \in \{1\} \\ y_2 \in \{2,3\} \\ y_3 \in \{2,3\} \\ y_4 \in \{5\} \\ y_5 \in \{4,6\} \end{array}$$

GAC achieved.

Disjunctive scheduling

Consider a disjunctive scheduling constraint:

$$\text{noOverlap}\left(\boxed{(s_1, s_2, s_3, s_5}), (p_1, p_2, p_3, p_5)\right)$$

Start time variables

Job j	*Release time* r_j	*Dead-line* d_j	*Processing time* p_{Aj}	p_{Bj}
1	0	10	1	5
2	0	10	3	6
3	2	7	3	7
4	2	10	4	6
5	4	7	2	5

Edge finding for disjunctive scheduling

Consider a disjunctive scheduling constraint:

$$\text{noOverlap}\big((s_1, s_2, s_3, s_5), \boxed{(p_1, p_2, p_3, p_5}\,)\big)$$

Processing times

Job j	Release time r_j	Dead-line d_j	Processing time p_{Aj}	p_{Bj}
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1	0	10	1	5
2	0	10	3	6
3	2	7	3	7
4	2	10	4	6
5	4	7	2	5

Variable domains defined by time windows and processing times

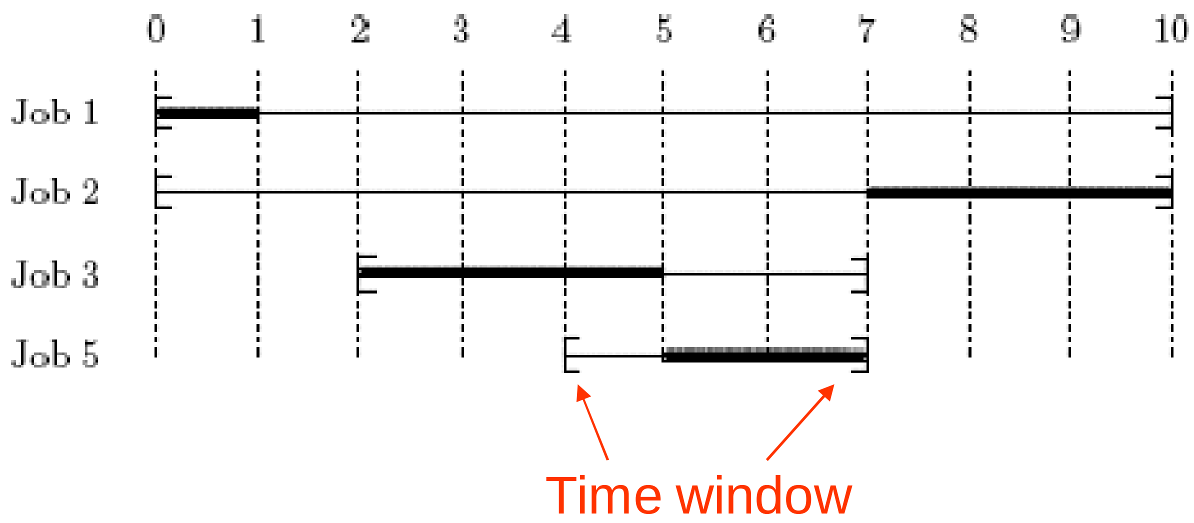
$$s_1 \in [0, 10-1]$$
$$s_2 \in [0, 10-3]$$
$$s_3 \in [2, 7-3]$$
$$s_5 \in [4, 7-2]$$

Edge finding for disjunctive scheduling

Consider a disjunctive scheduling constraint:

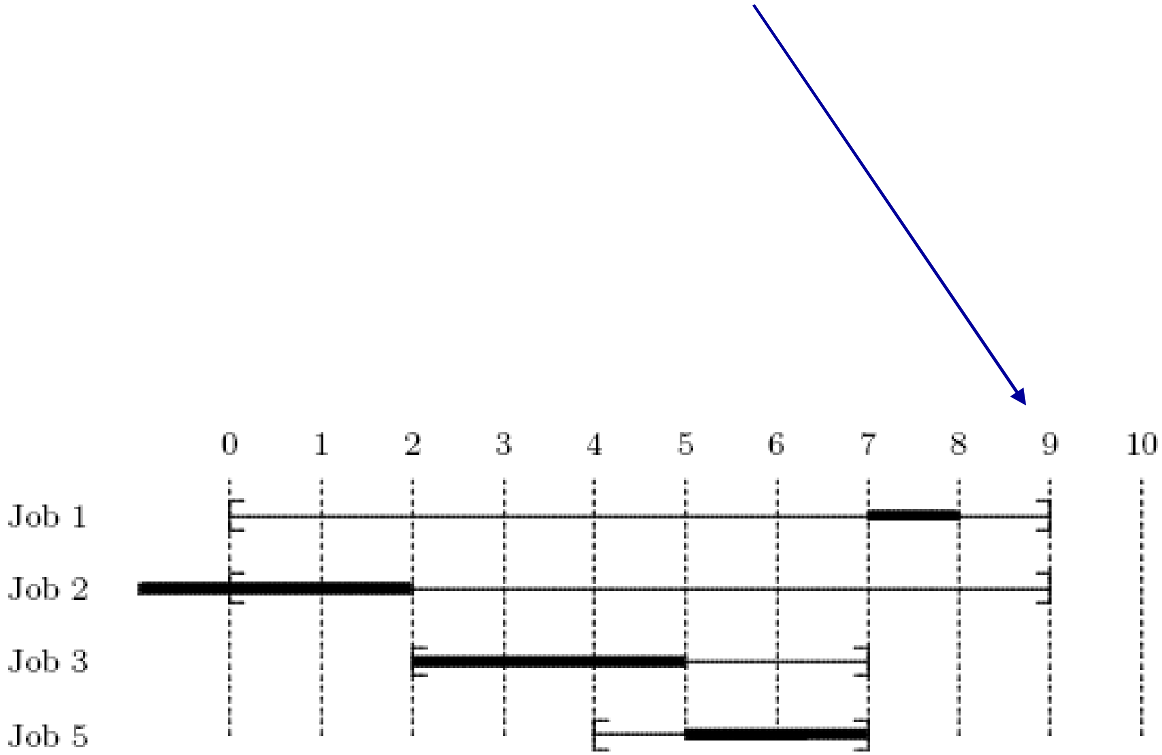
$$\text{noOverlap}\left((s_1, s_2, s_3, s_5), (p_1, p_2, p_3, p_5)\right)$$

A feasible (min makespan) solution:



Edge finding for disjunctive scheduling

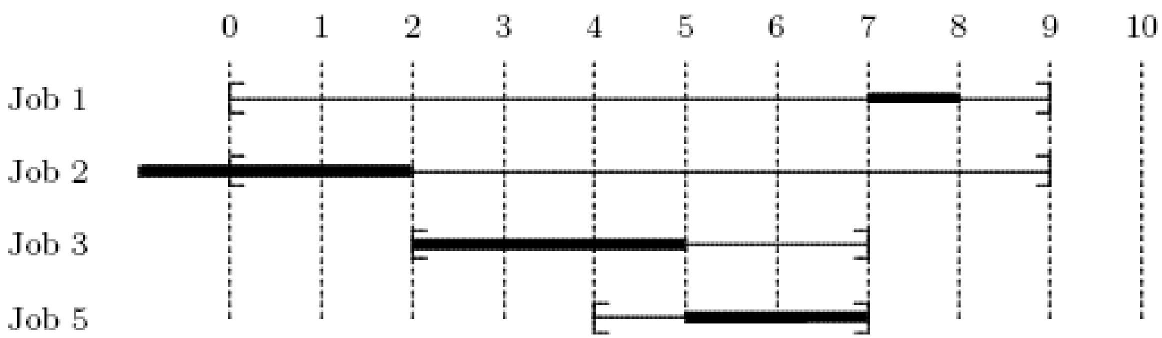
But let's reduce 2 of the deadlines to 9:



Edge finding for disjunctive scheduling

But let's reduce 2 of the deadlines to 9:

We will use edge finding to prove that there is no feasible schedule.

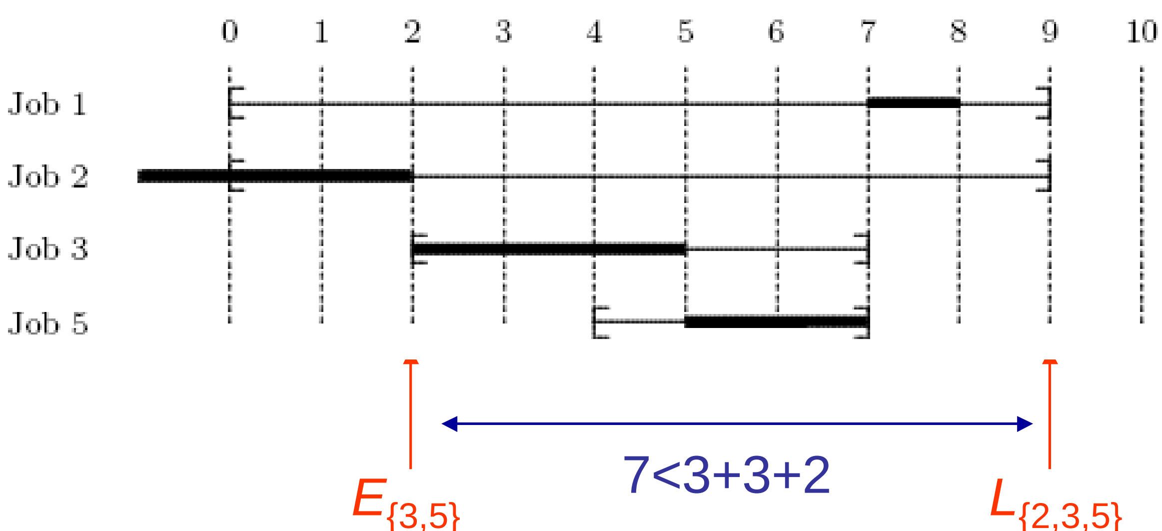


Edge finding for disjunctive scheduling

We can deduce that job 2 must precede jobs 3 and 5: $2 \ll \{3,5\}$

Because if job 2 is not first, there is not enough time for all 3 jobs within the time windows:

$$L_{\{2,3,5\}} - E_{\{3,5\}} < p_{\{2,3,5\}}$$



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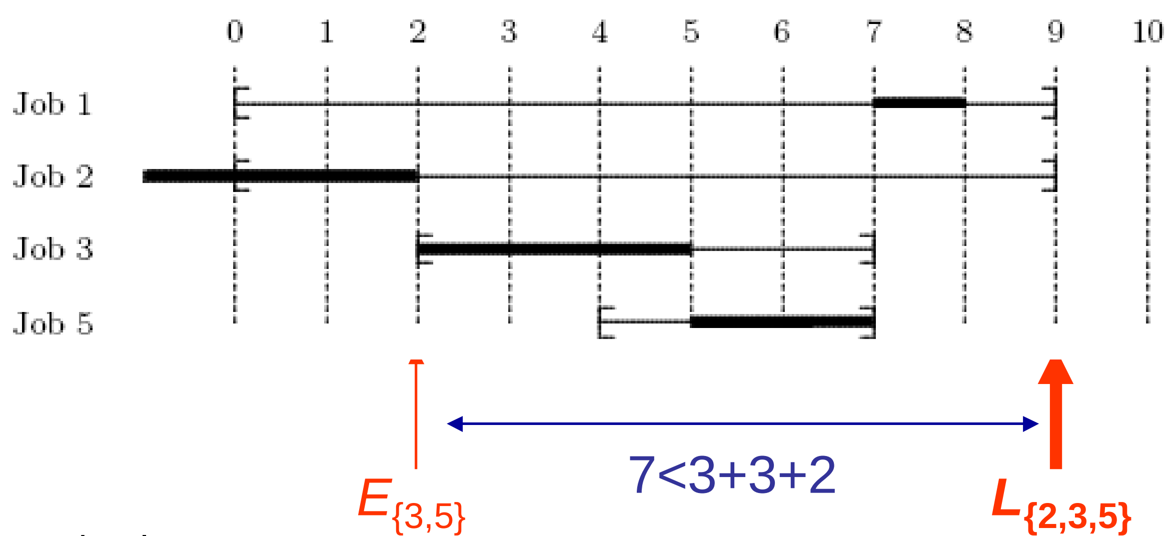
Latest deadline



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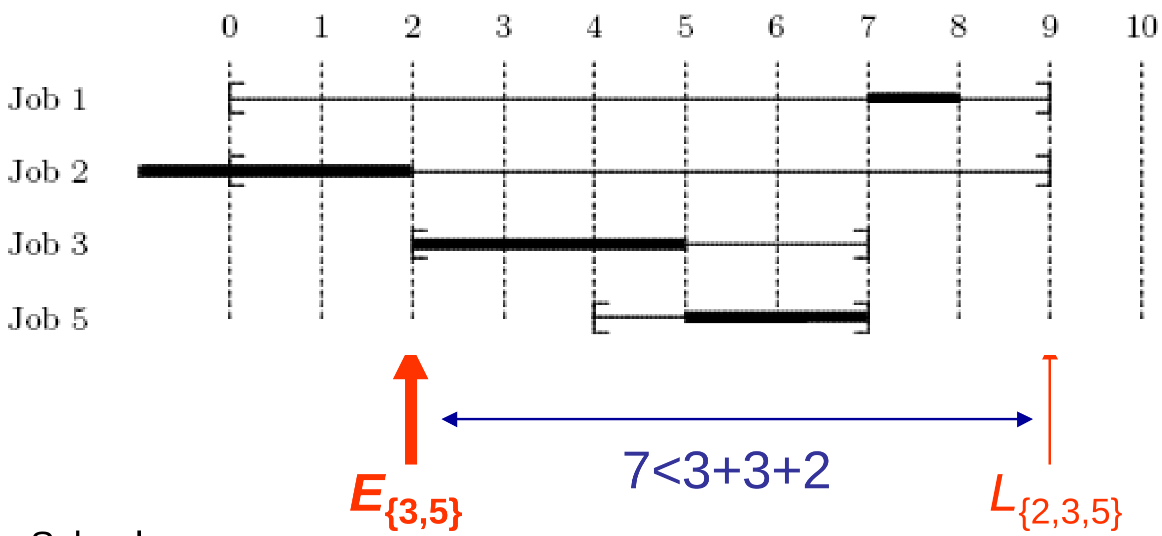
$$L_{\{2,3,5\}} - \boxed{E_{\{3,5\}}} < p_{\{2,3,5\}}$$

Earliest release time



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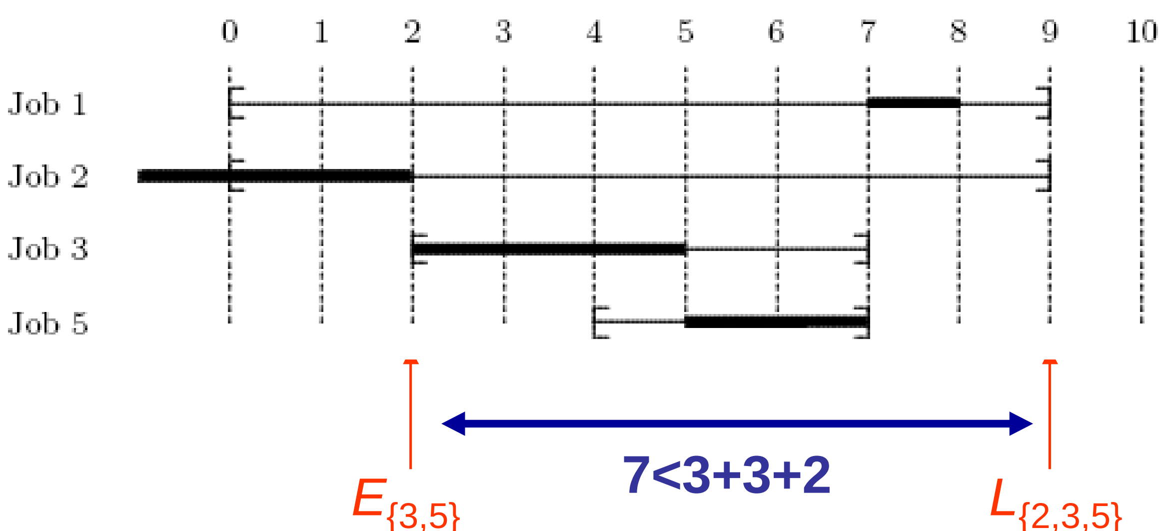
Total processing time



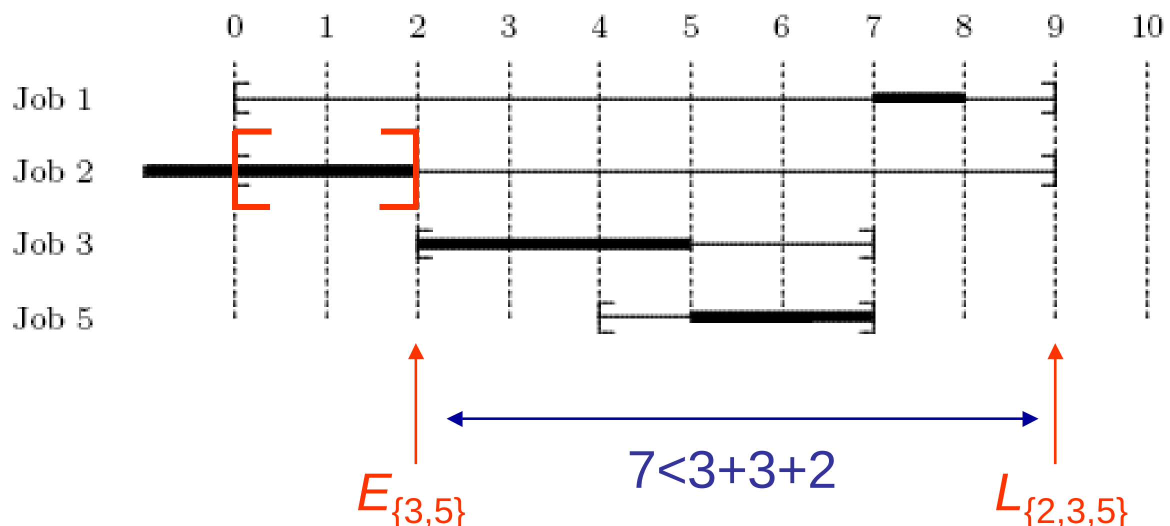
Edge finding for disjunctive scheduling

We can deduce that job 2 must precede jobs 3 and 5: $2 \ll \{3,5\}$

So we can tighten deadline of job 2 to minimum of

$$L_{\{3\}} - p_{\{3\}} = 4 \qquad L_{\{5\}} - p_{\{5\}} = 5 \qquad L_{\{3,5\}} - p_{\{3,5\}} = 2$$

Since time window of job 2 is now too narrow, there is no feasible schedule.



Edge finding for disjunctive scheduling

In general, we can deduce that job *k* must precede all the jobs in set *J*: $k \ll J$

If there is not enough time for all the jobs after the earliest release time of the jobs in *J*

$$L_{J \cup \{k\}} - E_J < p_{J \cup \{k\}} \qquad L_{\{2,3,5\}} - E_{\{3,5\}} < p_{\{2,3,5\}}$$

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$$L_{J\cup\{k\}} - E_J < p_{J\cup\{k\}} \qquad L_{\{2,3,5\}} - E_{\{3,5\}} < p_{\{2,3,5\}}$$

Now we can tighten the deadline for job *k* to:

$$\min_{J'\subset J}\left\{L_{J'} - p_{J'}\right\} \qquad L_{\{3,5\}} - p_{\{3,5\}} = 2$$

Edge finding for disjunctive scheduling

There is a symmetric rule: $k \gg J$

If there is not enough time for all the jobs before the latest deadline of the jobs in *J:*

$$L_J - E_{J\cup\{k\}} < p_{J\cup\{k\}}$$

Now we can tighten the release date for job *k* to*:*

$$\max_{J'\subset J}\left\{E_{J'} + p_{J'}\right\}$$

Edge finding for disjunctive scheduling

Problem: how can we avoid enumerating all subsets *J* of jobs to find edges?

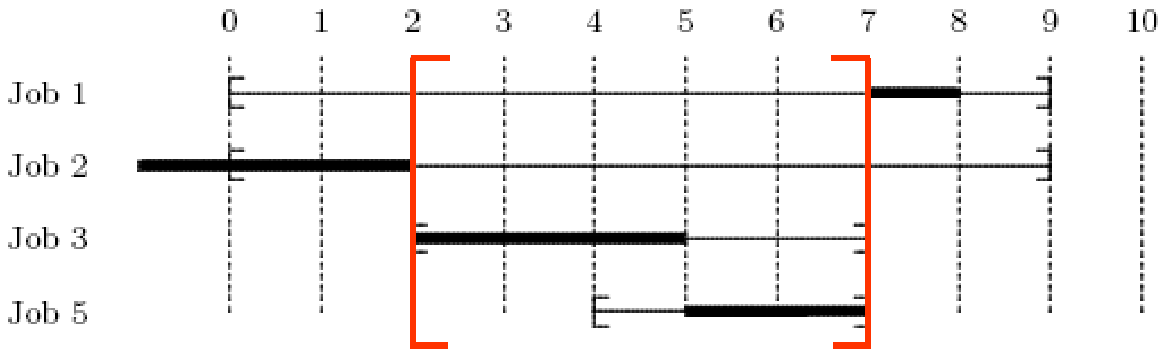
$$L_{J\cup\{k\}} - E_J < p_{J\cup\{k\}}$$

…and all subsets *J´* of *J* to tighten the bounds?

$$\min_{J'\subset J}\left\{L_{J'} - p_{J'}\right\}$$

Edge finding for disjunctive scheduling

Key result: We only have to consider sets *J* whose time windows lie within some interval.



e.g., $J = \{3,5\}$

Edge finding for disjunctive scheduling

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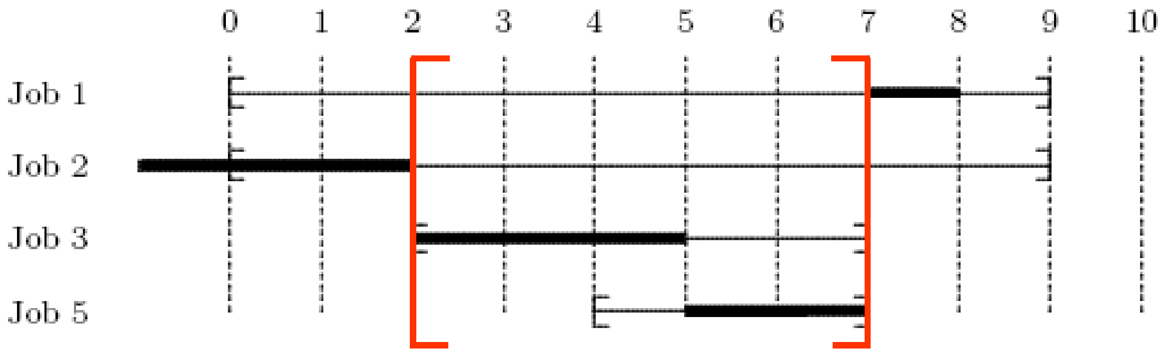


e.g., $J = \{3,5\}$

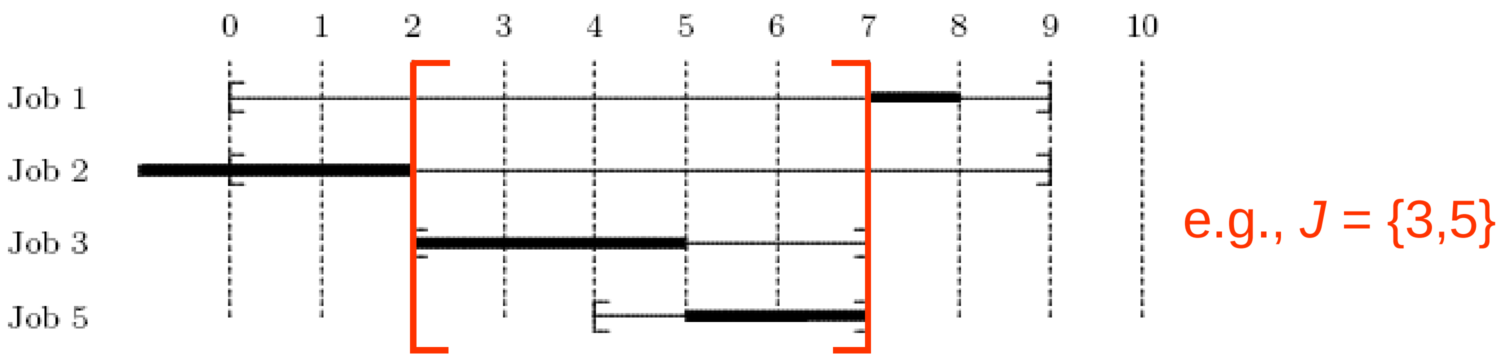
Removing a job from those within an interval only weakens the test

$$L_{J\cup\{k\}} - E_J < p_{J\cup\{k\}}$$

There are a polynomial number of intervals defined by release times and deadlines.

Edge finding for disjunctive scheduling

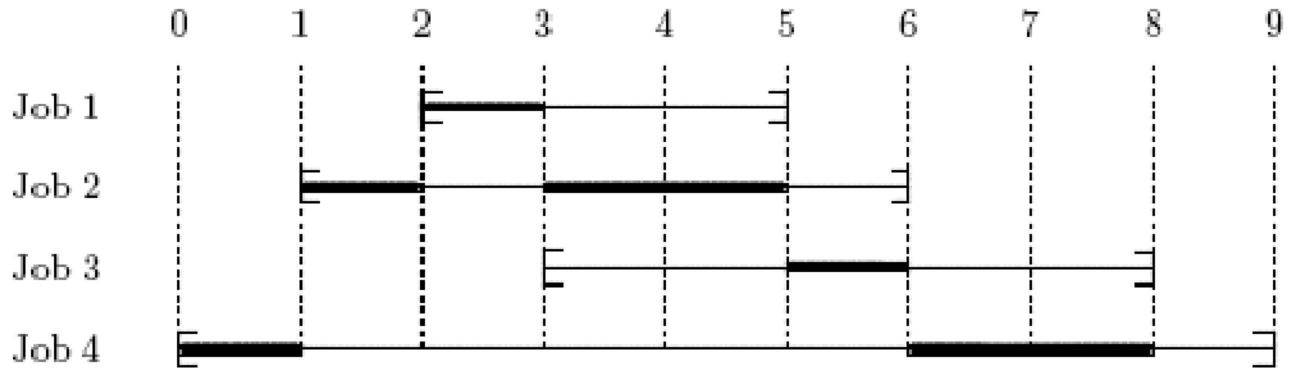
Key result: We only have to consider sets *J* whose time windows lie within some interval.



Note: Edge finding does not achieve bounds consistency, which is an NP-hard problem.

Edge finding for disjunctive scheduling

One $O(n^2)$ algorithm is based on the Jackson pre-emptive schedule (JPS). Using a different example, the JPS is:



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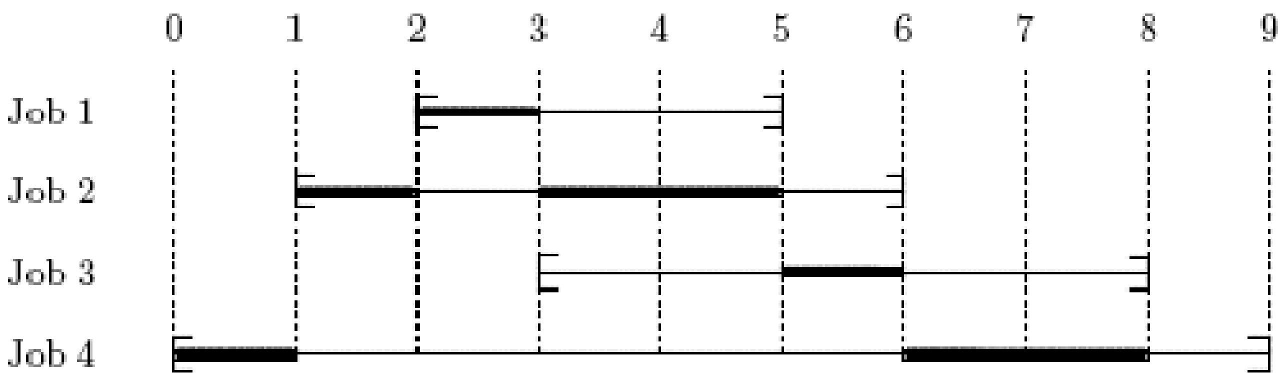


Jobs unfinished at time E_i in JPS

For each job i

Scan jobs $k \in \boxed{J_i}$ in decreasing order of L_k

Select first k for which $L_k - E_i < p_i + \boxed{\bar{p}_{J_{ik}}}$

Total remaining processing time in JPS of jobs in J_{ik}

Conclude that $i \gg \boxed{J_{ik}}$

Jobs $j \neq i$ in J_i with $L_j \leq L_k$

Update E_i to $\boxed{\text{JPS}(i,k)}$

Latest completion time in JPS of jobs in J_{ik}

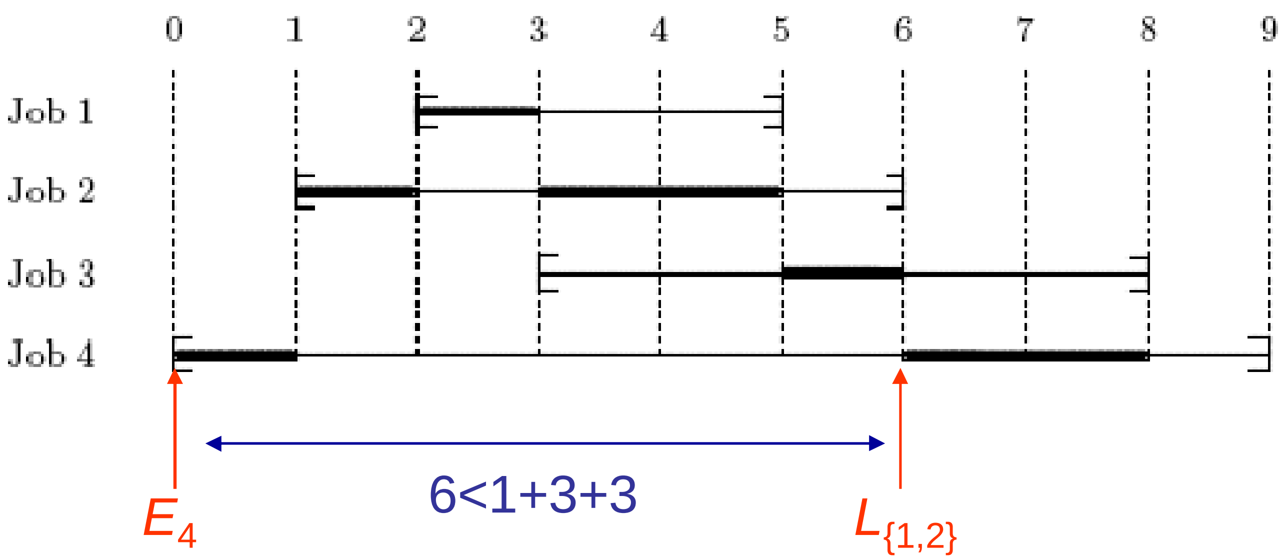
Not-first/not-last rules

We can deduce that job 4 cannot precede jobs 1 and 2:

$$\neg\left(4 \ll \{1,2\}\right)$$

Because if job 4 is first, there is too little time to complete the jobs before the later deadline of jobs 1 and 2:

$$L_{\{1,2\}} - E_4 < p_1 + p_2 + p_4$$



Not-first/not-last rules

We can deduce that job 4 cannot precede jobs 1 and 2:

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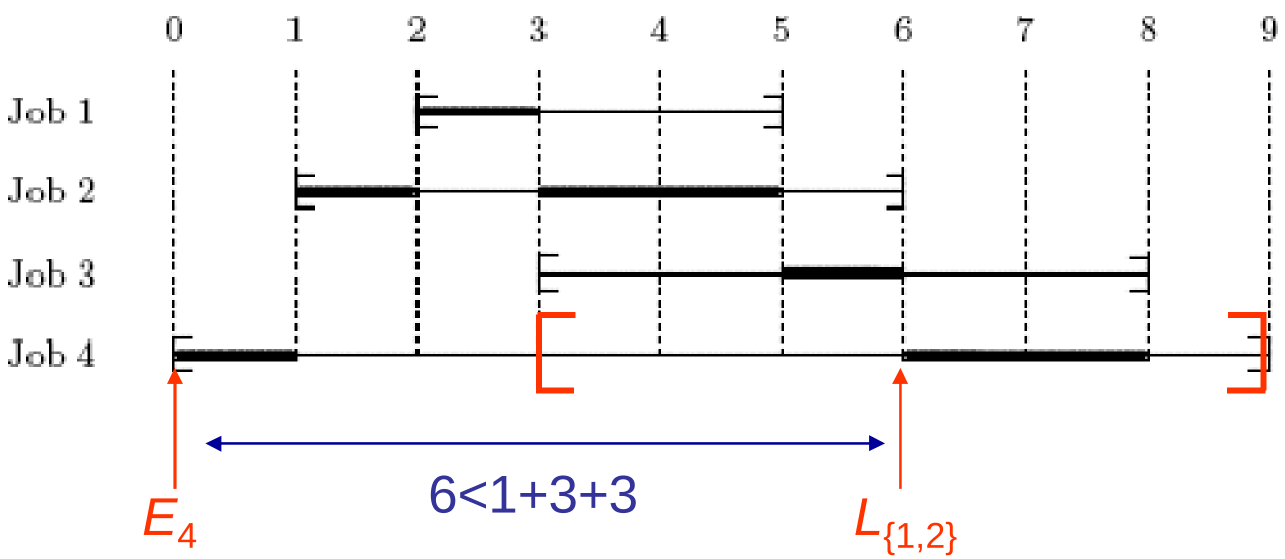
Now we can tighten the release time of job 4 to minimum of:

$$E_1 + p_1 = 3 \qquad E_2 + p_2 = 4$$



Not-first/not-last rules

In general, we can deduce that job *k* cannot precede all the jobs in *J*:

$$\neg\left(k \ll J\right)$$

if there is too little time after release time of job *k* to complete all jobs before the latest deadline in *J*:

$$L_J - E_k < p_J$$

Now we can update E_i to

$$\min_{j \in J}\left\{E_j + p_j\right\}$$

Not-first/not-last rules

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There is a symmetric not-last rule.

The rules can be applied in polynomial time, although an efficient algorithm is quite complicated.

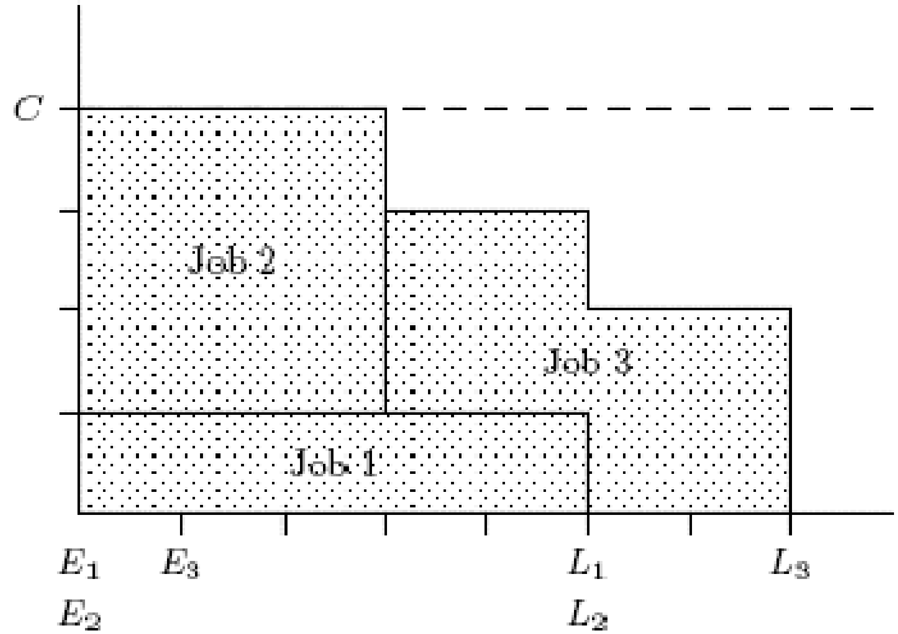
Cumulative scheduling

Consider a cumulative scheduling constraint:

$$\text{cumulative}\big((s_1,s_2,s_3),(p_1,p_2,p_3),(c_1,c_2,c_3),C\big)$$

j	p_j	c_j	E_j	L_j
1	5	1	0	5
2	3	3	0	5
3	4	2	1	7

A feasible solution:

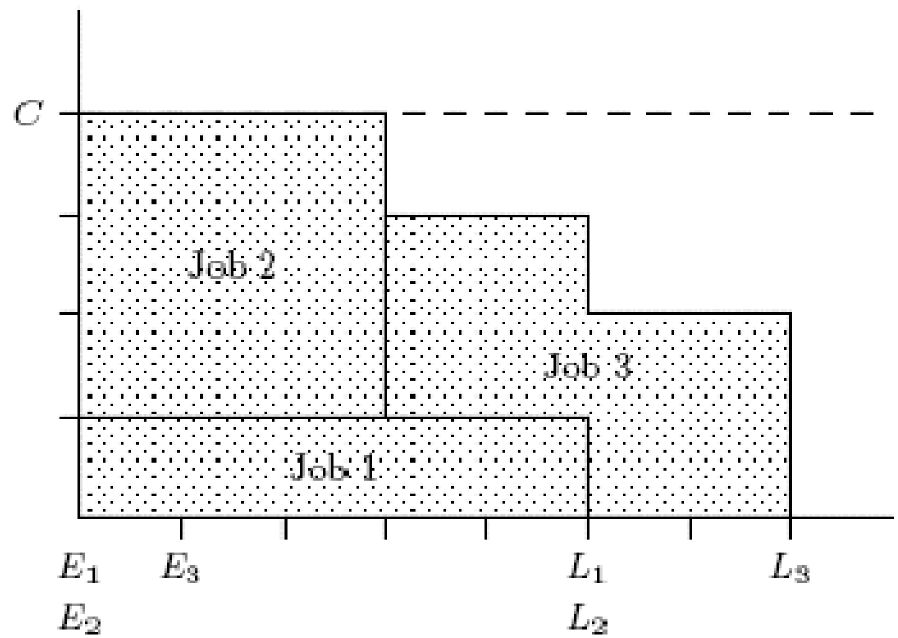


Edge finding for cumulative scheduling

We can deduce that job 3 must finish after the others finish: $3 > \{1,2\}$

Because the total **energy** required exceeds the area between the earliest release time and the later deadline of jobs 1,2:

$$e_3 + e_{\{1,2\}} > C \cdot \left(L_{\{1,2\}} - E_{\{1,2,3\}} \right)$$



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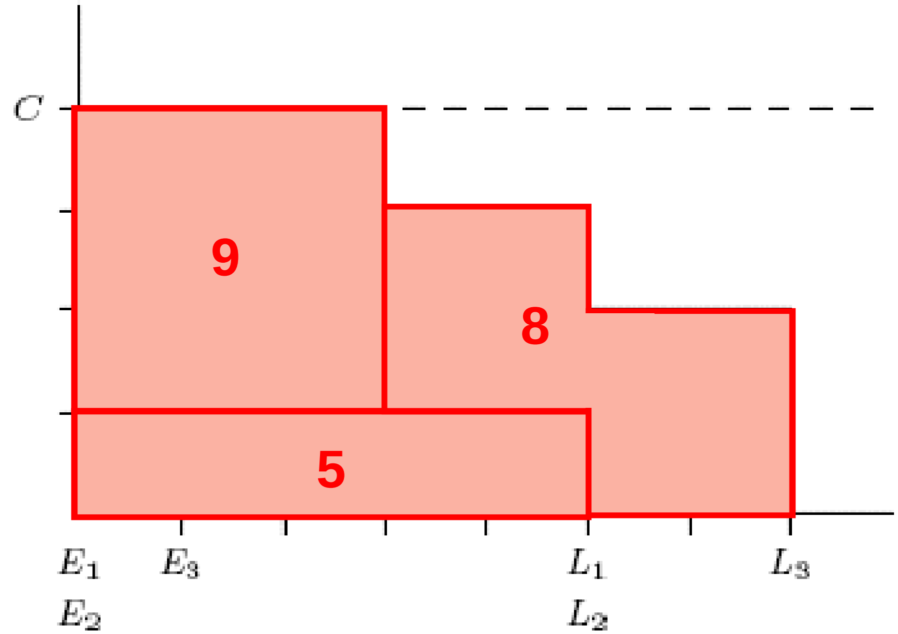
Total energy required = 22



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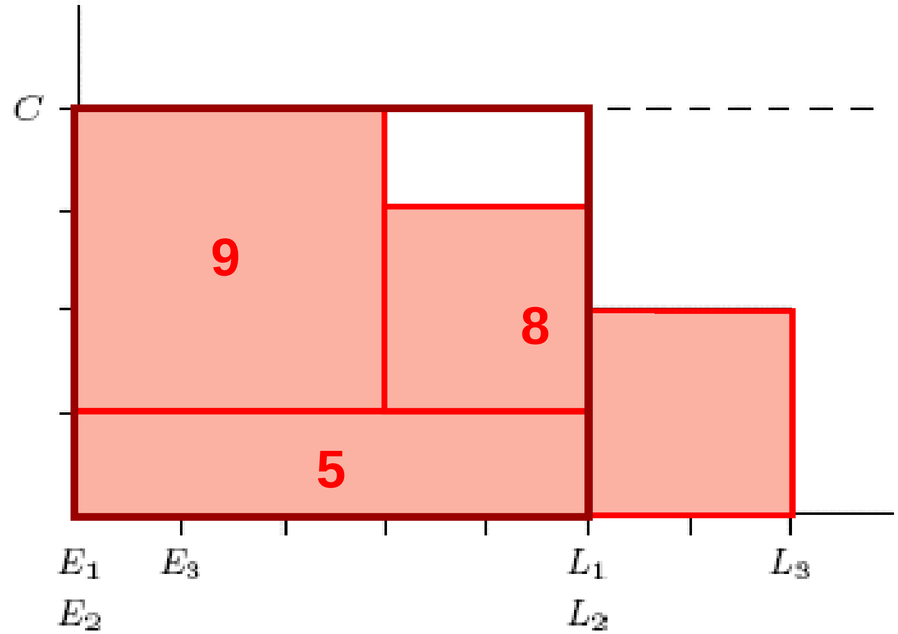
Total energy required = 22

Area available = 20



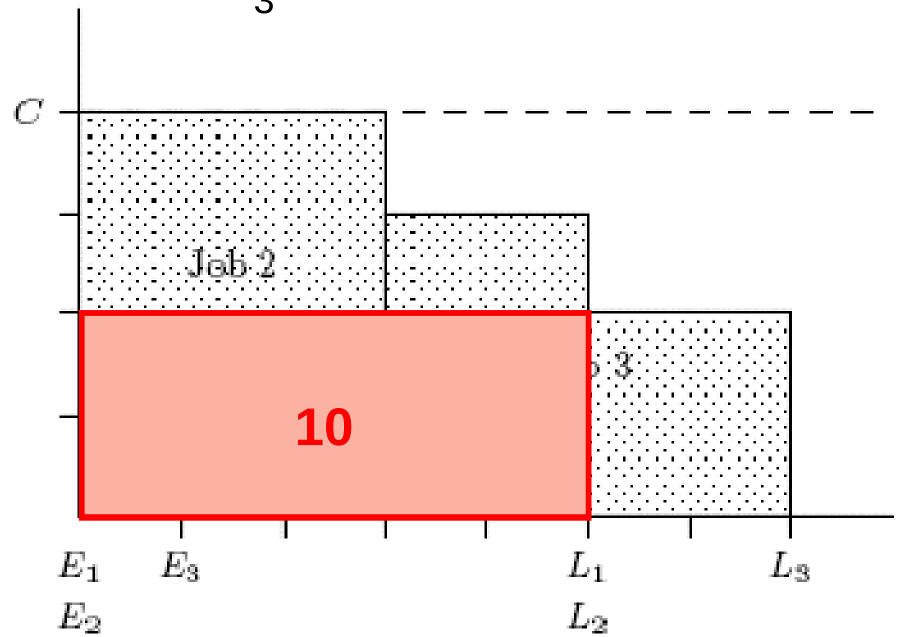
Edge finding for cumulative scheduling

We can deduce that job 3 must finish after the others finish: $3 > \{1,2\}$

We can update the release time of job 3 to

$$E_{\{1,2\}} + \frac{e_J - \boxed{(C - c_3)(L_{\{1,2\}} - E_{\{1,2\}})}}{c_3}$$

Energy available for jobs 1,2 if space is left for job 3 to start anytime = 10



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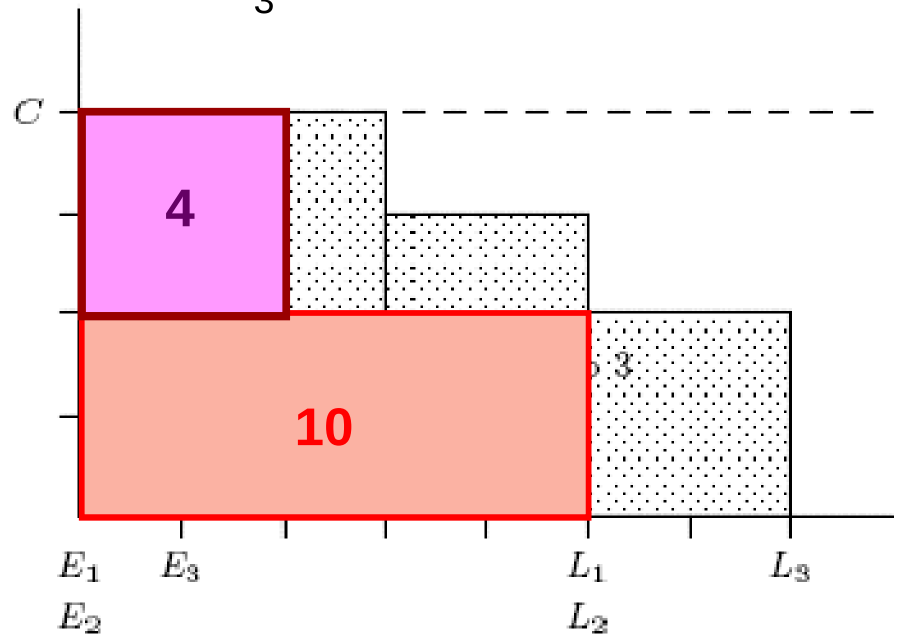
Energy available for jobs 1,2 if space is left for job 3 to start anytime = 10

Excess energy required by jobs 1,2 = 4



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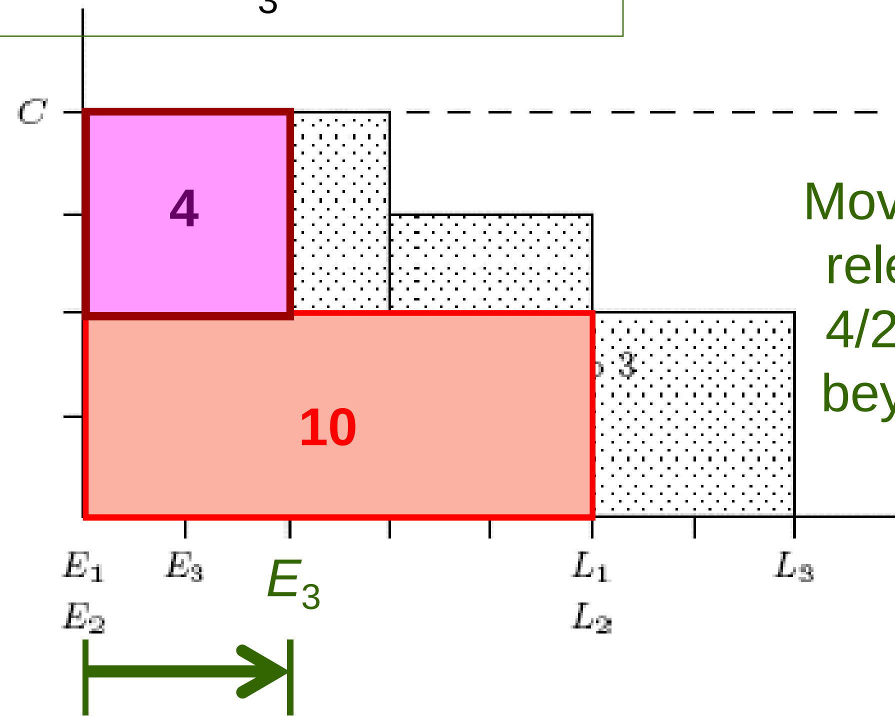


Move up job 3 release time 4/2 = 2 units beyond $E_{\{1,2\}}$

Edge finding for cumulative scheduling

In general, if $e_{J\cup\{k\}} > C\cdot\left(L_J - E_{J\cup\{k\}}\right)$

then $k > J$, and update E_k to

$$\max_{\substack{J'\subset J \\ e_{J'}-(C-c_k)(L_{J'}-E_{J'})>0}} \left\{ E_{J'} + \frac{e_{J'} - (C - c_k)(L_{J'} - E_{J'})}{c_k} \right\}$$

In general, if $e_{J\cup\{k\}} > C\cdot\left(L_{J\cup\{k\}} - E_J\right)$

then $k < J$, and update L_k to

$$\min_{\substack{J'\subset J \\ e_{J'}-(C-c_k)(L_{J'}-E_{J'})>0}} \left\{ L_{J'} - \frac{e_{J'} - (C - c_k)(L_{J'} - E_{J'})}{c_k} \right\}$$

Edge finding for cumulative scheduling

There is an $O(n^2)$ algorithm that finds all applications of the edge finding rules.

Other propagation rules for cumulative scheduling

- Extended edge finding.
- Timetabling.
- Not-first/not-last rules.
- Energetic reasoning.

Linear Relaxation

Why Relax?

Algebraic Analysis of LP

Linear Programming Duality

LP-Based Domain Filtering

Example: Single-Vehicle Routing

Disjunctions of Linear Systems

Why Relax?

Solving a relaxation of a problem can:

- Tighten variable bounds.
- Possibly solve original problem.
- Guide the search in a promising direction.
- Filter domains using reduced costs or Lagrange multipliers.
- Prune the search tree using a bound on the optimal value.
- Provide a more global view, because a single OR relaxation can pool relaxations of several constraints.

Some MP models that can provide relaxations:

- Linear programming (LP).
- Mixed integer linear programming (MILP)
  - Can itself be relaxed as an LP.
  - LP relaxation can be strengthened with cutting planes.
- Lagrangean relaxation.
- Specialized relaxations.
  - For particular problem classes.
  - For global constraints.

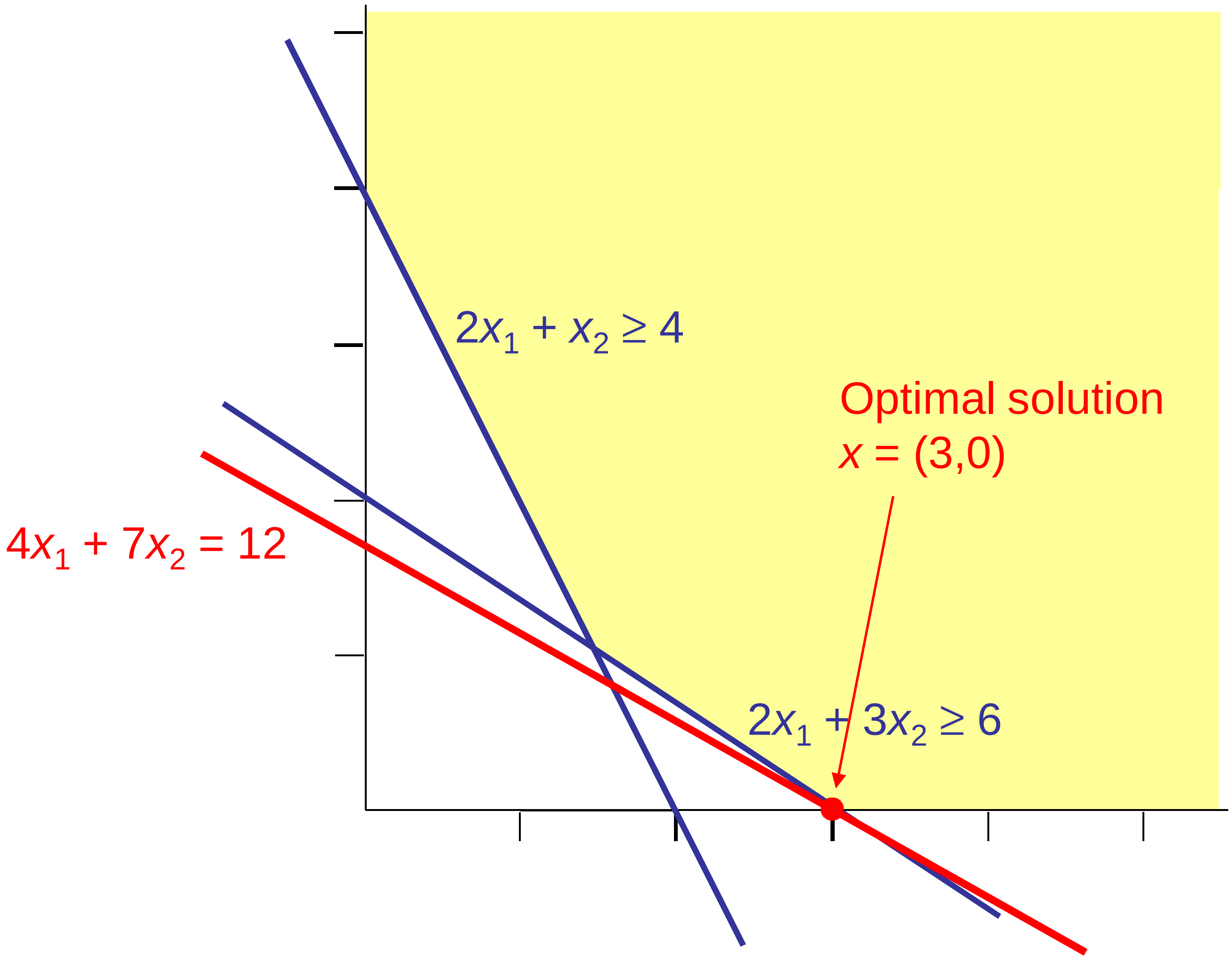
Motivation

- **Linear programming** is remarkably versatile for representing real-world problems.
- LP is by far the most widely used tool for **relaxation**.
- LP relaxations can be strengthened by **cutting planes.**
  - Based on polyhedral analysis.
- LP has an elegant and powerful **duality theory**.
  - Useful for domain filtering, and much else.
- The LP problem is **extremely well solved**.

Algebraic Analysis of LP

An example…

$$\min \;\; 4x_1 + 7x_2$$
$$2x_1 + 3x_2 \geq 6$$
$$2x_1 + x_2 \geq 4$$
$$x_1, x_2 \geq 0$$



Algebraic Analysis of LP

Rewrite

$$\begin{aligned}&\min\ 4x_1 + 7x_2\\&2x_1 + 3x_2 \ge 6\\&2x_1 + x_2 \ge 4\\&x_1, x_2 \ge 0\end{aligned}$$

as

$$\begin{aligned}&\min\ 4x_1 + 7x_2\\&2x_1 + 3x_2 - x_3 = 6\\&2x_1 + x_2 - x_4 = 4\\&x_1, x_2, x_3, x_4 \ge 0\end{aligned}$$

In general an LP has the form

$$\begin{aligned}&\min\ cx\\&Ax = b\\&x \ge 0\end{aligned}$$

Algebraic analysis of LP

Write

$$\begin{aligned} &\min cx \\ &\boxed{A}x = b \\ &x \geq 0 \end{aligned}$$

$m \times n$ matrix

as

$$\begin{aligned} &\min c_B x_B + c_N x_N \\ &Bx_B + Nx_N = b \\ &\boxed{x_B}, \boxed{x_N} \geq 0 \end{aligned}$$

Basic variables

Nonbasic variables

where

$$A = [\boxed{B}\ N]$$

Any set of m linearly independent columns of A.

These form a **basis** for the space spanned by the columns.

Algebraic analysis of LP

Write

$$\begin{aligned} &\min cx \\ &Ax = b \\ &x \geq 0 \end{aligned}$$

as

$$\begin{aligned} &\min c_B x_B + c_N x_N \\ &\boxed{Bx_B + Nx_N = b} \\ &x_B, x_N \geq 0 \end{aligned}$$

where

$$A = [B\ N]$$

Solve constraint equation for x_B: $x_B = B^{-1}b - B^{-1}N\boxed{x_N}$

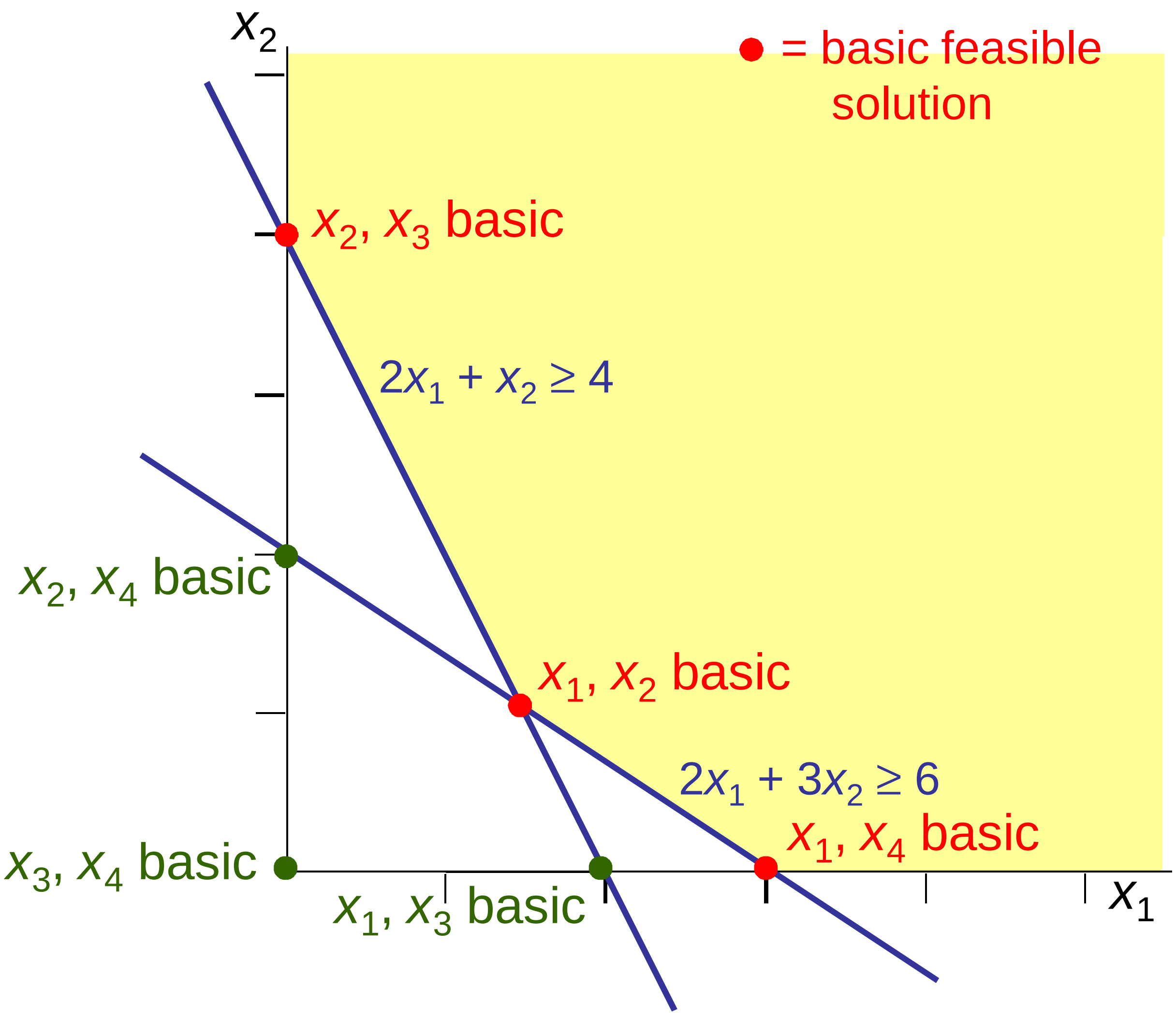
All solutions can be obtained by setting x_N to some value.

The solution is **basic** if $x_N = 0$.

It is a **basic feasible solution** if $x_N = 0$ and $x_B \geq 0$.

Example…

$$\min \ \ 4x_1 + 7x_2$$
$$2x_1 + 3x_2 - x_3 = 6$$
$$2x_1 + x_2 - x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$



Algebraic analysis of LP

Write $\min cx$ as $\min \boxed{c_B x_B + c_N x_N}$ where

$$Ax = b \qquad\qquad B x_B + N x_N = b \qquad\qquad A = [B\ N]$$

$$x \geq 0 \qquad\qquad x_B, x_N \geq 0$$

Solve constraint equation for x_B: $x_B = B^{-1}b - B^{-1}N x_N$

Express cost in terms of nonbasic variables:

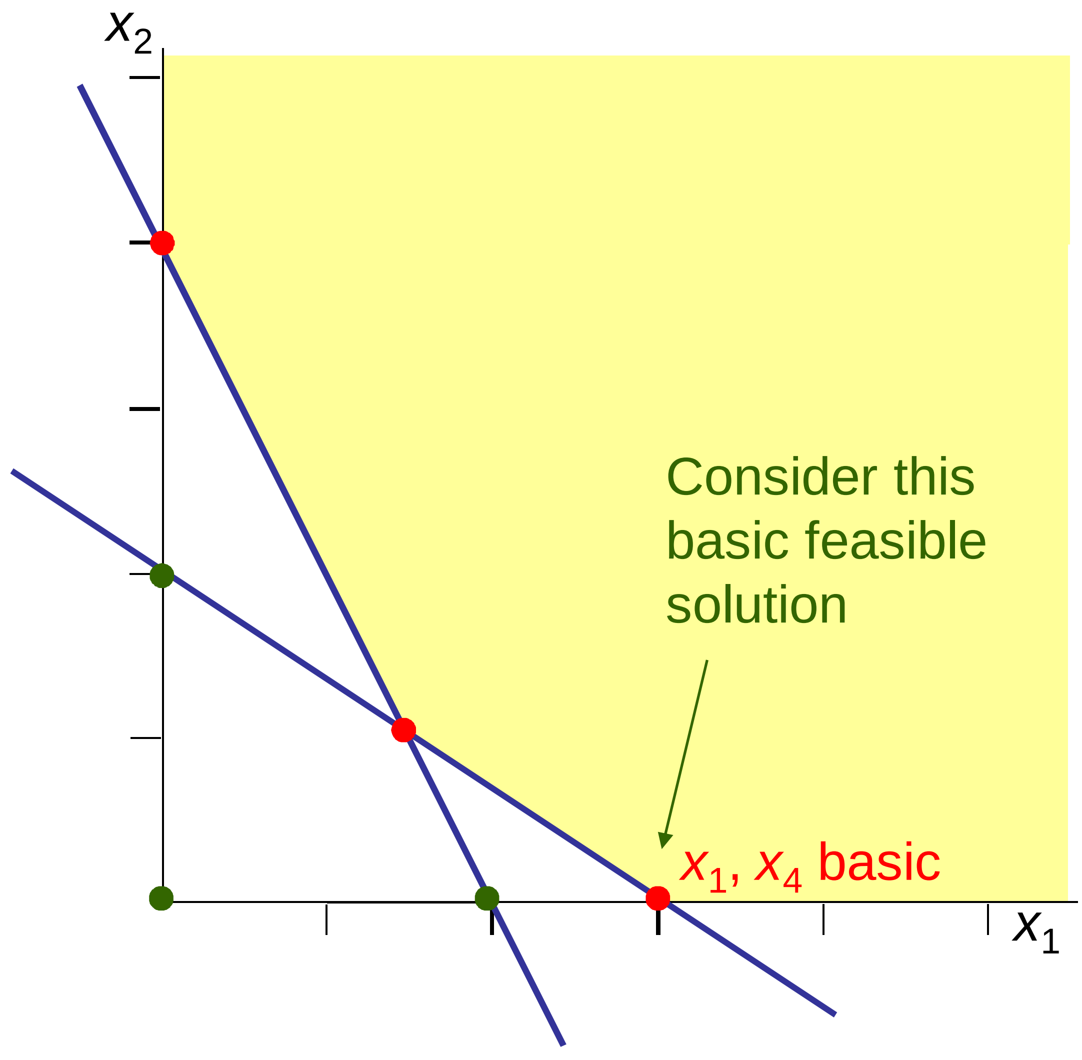
$$c_B B^{-1} b + (\boxed{c_N - c_B B^{-1} N}) x_N$$

Vector of reduced costs

Since $x_N \geq 0$, basic solution $(x_B,0)$ is optimal if reduced costs are nonnegative.

Example…

$$\min \ 4x_1 + 7x_2$$
$$2x_1 + 3x_2 - x_3 = 6$$
$$2x_1 + x_2 - x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$



Example…

Write…

$$\min \ 4x_1 + 7x_2$$
$$2x_1 + 3x_2 - x_3 = 6$$
$$2x_1 + x_2 - x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

as…

$$\min \ \overbrace{\boxed{[4 \ \ 0]\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}}}^{c_B x_B} + \overbrace{\boxed{[7 \ \ 0]\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}}}^{c_N x_N}$$

$$B x_B \ \boxed{\begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}} + \underbrace{\boxed{\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}}}_{N x_N} = \underbrace{\boxed{\begin{bmatrix} 6 \\ 4 \end{bmatrix}}}_{b}$$

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example…

$$\min \overbrace{\boxed{[4\ \ 0]\begin{bmatrix}x_1\\ x_4\end{bmatrix}}}^{c_Bx_B} + \overbrace{\boxed{[7\ \ 0]\begin{bmatrix}x_2\\ x_3\end{bmatrix}}}^{c_Nx_N}$$

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$$\begin{bmatrix}x_1\\ x_4\end{bmatrix}, \begin{bmatrix}x_1\\ x_4\end{bmatrix} \geq \begin{bmatrix}0\\ 0\end{bmatrix}$$

Example…

$$c_Bx_B \qquad c_Nx_N$$

$$\min \boxed{\begin{bmatrix}4 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_4\end{bmatrix}} + \boxed{\begin{bmatrix}7 & 0\end{bmatrix}\begin{bmatrix}x_2\\x_3\end{bmatrix}}$$

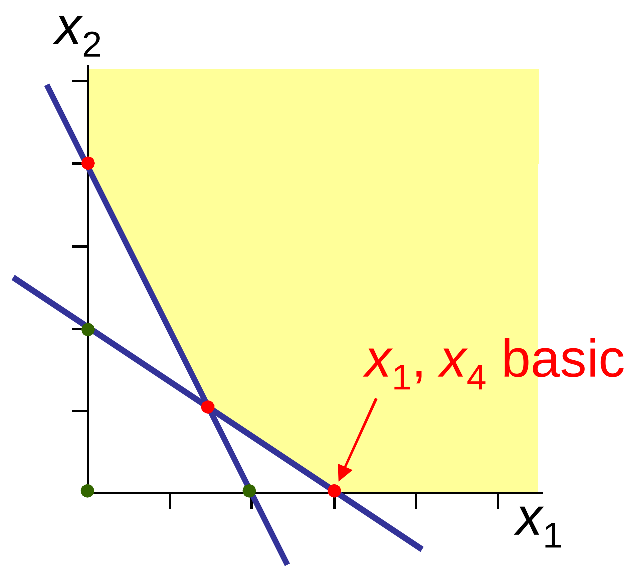
$$Bx_B \ \boxed{\begin{bmatrix}2 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_4\end{bmatrix}} + \boxed{\begin{bmatrix}3 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_4\end{bmatrix}} = \boxed{\begin{bmatrix}6\\4\end{bmatrix}}$$

$$\begin{bmatrix}x_1\\x_4\end{bmatrix}, \begin{bmatrix}x_1\\x_4\end{bmatrix} \geq \begin{bmatrix}0\\0\end{bmatrix} \qquad Nx_N \qquad b$$

Basic solution is

$$x_B = B^{-1}b - B^{-1}Nx_N = B^{-1}b$$

$$= \begin{bmatrix}x_1\\x_4\end{bmatrix} = \begin{bmatrix}1/2 & 0\\1 & -1\end{bmatrix}\begin{bmatrix}6\\4\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$



Example…

$$\min \boxed{[4\ \ 0]\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}}^{c_B x_B} + \boxed{[7\ \ 0]\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}}^{c_N x_N}$$

$$Bx_B\ \boxed{\begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}} + \underbrace{\boxed{\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}}}_{Nx_N} = \boxed{\begin{bmatrix} 6 \\ 4 \end{bmatrix}}$$

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Basic solution is

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Reduced costs are

$$c_N - c_B B^{-1} N$$

$$= [7 \quad 0] - [4 \quad 0]\begin{bmatrix} 1/2 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= [1 \quad 2] \geq [0 \quad 0]$$

Solution is optimal

Linear Programming Duality

An LP can be viewed as an inference problem…

$$\min\ cx \quad = \quad \max\ v$$
$$Ax \geq b \qquad\qquad Ax \geq b \overset{x\geq 0}{\Rightarrow} cx \geq v$$
$$x \geq 0$$

implies

Dual problem: Find the tightest lower bound on the objective function that is implied by the constraints.

An LP can be viewed as an inference problem…

$$\begin{array}{l} \min\ cx \\ Ax \ge b \\ x \ge 0 \end{array} \quad = \quad \begin{array}{l} \max\ v \\ Ax \ge b \overset{x\ge 0}{\Rightarrow} cx \ge v \end{array}$$

That is, some **surrogate** (nonnegative linear combination) of $Ax \ge b$ dominates $cx \ge v$

From Farkas Lemma: If $Ax \ge b$, $x \ge 0$ is feasible,

$$Ax \ge b \overset{x\ge 0}{\Rightarrow} cx \ge v \quad \text{iff} \quad \lambda Ax \ge \lambda b \ \boxed{\text{dominates}}\ cx \ge v$$

for some $\lambda \ge 0$

$\lambda A \le c$ and $\lambda b \ge v$

An LP can be viewed as an inference problem…

$$\begin{array}{l} \min\ cx \\ Ax \geq b \\ x \geq 0 \end{array} \quad = \quad \begin{array}{l} \max\ v \\ Ax \geq b \overset{x\geq 0}{\Rightarrow} cx \geq v \end{array} \quad = \quad \begin{array}{l} \max\ \lambda b \\ \lambda A \leq c \\ \lambda \geq 0 \end{array}$$

This is the **classical LP dual**

From Farkas Lemma: If $Ax \geq b$, $x \geq 0$ is feasible,

$Ax \geq b \overset{x\geq 0}{\Rightarrow} cx \geq v$ iff $\lambda Ax \geq \lambda b$ dominates $cx \geq v$ for some $\lambda \geq 0$

$\lambda A \leq c$ and $\lambda b \geq v$

This equality is called **strong duality.**

$$\min\ cx \quad = \quad \max\ \lambda b$$
$$Ax \geq b \qquad\qquad \lambda A \leq c$$
$$x \geq 0 \qquad\qquad \lambda \geq 0$$

This is the **classical LP dual**

If $Ax \geq b$, $x \geq 0$ is feasible

Note that the dual of the dual is the **primal** (i.e., the original LP).

Example

Primal		*Dual*	
$\min\ 4x_1 + 7x_2$	$=$	$\max\ 6\lambda_1 + 4\lambda_2$	$= 12$
$2x_1 + 3x_2 \geq 6$	(λ_1)	$2\lambda_1 + 2\lambda_2 \leq 4$	(x_1)
$2x_1 + x_2 \geq 4$	(λ_1)	$3\lambda_1 + \lambda_2 \leq 7$	(x_2)
$x_1, x_2 \geq 0$		$\lambda_1, \lambda_2 \geq 0$	

A dual solution is $(\lambda_1, \lambda_2) = (2,0)$

$$2x_1 + 3x_2 \geq 6 \quad \cdot(\lambda_1 = 2)$$
$$2x_1 + x_2 \geq 4 \quad \cdot(\lambda_2 = 0)$$

Dual multipliers

$$4x_1 + 6x_2 \geq 12$$

Surrogate

dominates

$$4x_1 + 7x_2 \geq 12$$

Tightest bound on cost

Weak Duality

If x* is feasible in the primal problem

$$\min\ cx$$
$$Ax \geq b$$
$$x \geq 0$$

and λ* is feasible in the dual problem

$$\max\ \lambda b$$
$$\lambda A \leq c$$
$$\lambda \geq 0$$

then $cx^* \geq \lambda^* b$.

This is because

$$cx^* \geq \lambda^* A x^* \geq \lambda^* b$$

λ^* is dual feasible and $x^* \geq 0$

x^* is primal feasible and $\lambda^* \geq 0$

Dual multipliers as marginal costs

Suppose we perturb the RHS of an LP (i.e., change the requirement levels):

$$\min\ cx$$
$$Ax \geq b + \Delta b$$
$$x \geq 0$$

The dual of the perturbed LP has the same constraints at the original LP:

$$\max\ \lambda(b + \Delta b)$$
$$\lambda A \leq c$$
$$\lambda \geq 0$$

So an optimal solution λ^* of the original dual is feasible in the perturbed dual.

Dual multipliers as marginal costs

Suppose we perturb the RHS of an LP (i.e., change the requirement levels):

$$\min \ cx$$
$$Ax \ge b + \Delta b$$
$$x \ge 0$$

By weak duality, the optimal value of the perturbed LP is at least $\lambda^*(b + \Delta b) = \boxed{\lambda^* b} + \lambda^* \Delta b$.

Optimal value of original LP, by strong duality.

So λ_i^* is a lower bound on the marginal cost of increasing the i-th requirement by one unit ($\Delta b_i = 1$).

If $\lambda_i^* > 0$, the i-th constraint must be tight **(complementary slackness).**

Dual of an LP in equality form

Primal

$$\min c_B x_B + c_N x_N$$
$$B x_B + N x_N = b \qquad (\lambda)$$
$$x_B, x_N \geq 0$$

Dual

$$\max \lambda b$$
$$\lambda B \leq c_B \qquad (x_B)$$
$$\lambda N \leq c_N \qquad (x_B)$$
$$\lambda \text{ unrestricted}$$

Dual of an LP in equality form

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$$\max \lambda b$$
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$$\lambda N \leq c_N \qquad (x_B)$$
$$\lambda \text{ unrestricted}$$

Recall that reduced cost vector is $c_N - \underbrace{\boxed{c_B B^{-1}}}_{\lambda} N = c_N - \lambda N$

this solves the dual
if (x_B,0) solves the primal

Dual of an LP in equality form

Primal

$$\min c_B x_B + c_N x_N$$
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this solves the dual
if (x_B,0) solves the primal

Check:

$$\lambda B = c_B B^{-1} B = c_B$$
$$\lambda N = c_B B^{-1} N \leq c_N$$

Because reduced cost is nonnegative
at optimal solution (x_B,0).

Dual of an LP in equality form

Primal

$$\min c_B x_B + c_N x_N$$
$$Bx_B + Nx_N = b \qquad (\lambda)$$
$$x_B, x_N \geq 0$$

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$$\max \lambda b$$
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$$\lambda N \leq c_N \qquad (x_B)$$
$$\lambda \text{ unrestricted}$$

Recall that reduced cost vector is $c_N - \underbrace{\boxed{c_B B^{-1}}}_{\lambda} N = c_N - \lambda N$

this solves the dual
if $(x_B, 0)$ solves the primal

In the example,

$$\lambda = c_B B^{-1} = [4 \quad 0]\begin{bmatrix} 1/2 & 0 \\ 1 & -1 \end{bmatrix} = [2 \quad 0]$$

Dual of an LP in equality form

Primal

$$\min c_B x_B + c_N x_N$$
$$Bx_B + Nx_N = b \qquad (\lambda)$$
$$x_B, x_N \geq 0$$

Dual

$$\max \lambda b$$
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$$\lambda N \leq c_N \qquad (x_B)$$
$$\lambda \text{ unrestricted}$$

Recall that reduced cost vector is $c_N - \underbrace{\boxed{c_B B^{-1}}}_{\lambda} N = c_N - \lambda N$

Note that the reduced cost of an individual variable x_j is $r_j = c_j - \lambda \boxed{A_j}$

Column j of A

LP-based Domain Filtering

Let $\begin{array}{l} \min\ cx \\ Ax \geq b \\ x \geq 0 \end{array}$ be an LP relaxation of a CP problem.

- One way to filter the domain of x_j is to minimize and maximize x_j subject to $Ax \geq b$, $x \geq 0$.
  - This is time consuming.
- A faster method is to use **dual multipliers** to derive valid inequalities.
  - A special case of this method uses **reduced costs** to bound or fix variables.
  - **Reduced-cost variable fixing** is a widely used technique in OR.

Suspose:

$$\begin{aligned} \min\ & cx \\ Ax &\geq b \\ x &\geq 0 \end{aligned}$$

has optimal solution x^*, optimal value v^*, and optimal dual solution λ^*.

…and $\lambda_i^* > 0$, which means the i-th constraint is tight (complementary slackness);

…and the LP is a relaxation of a CP problem;

…and we have a feasible solution of the CP problem with value U, so that U is an upper bound on the optimal value.

Supposing $\begin{array}{l} \min\ cx \\ Ax \geq b \\ x \geq 0 \end{array}$ has optimal solution x^*, optimal value v^*, and optimal dual solution λ^*:

If x were to change to a value other than x^*, the LHS of i-th constraint $A^i x \geq b_i$ would change by some amount Δb_i.

Since the constraint is tight, this would increase the optimal value as much as changing the constraint to $A^i x \geq b_i + \Delta b_i$.

So it would increase the optimal value at least $\lambda_i^* \Delta b_i$.

Supposing $\begin{array}{l} \min \ cx \\ Ax \geq b \\ x \geq 0 \end{array}$ has optimal solution x^*, optimal value v^*, and optimal dual solution λ^*:

We have found: a change in x that changes $A^i x$ by Δb_i increases the optimal value of LP at least $\lambda_i^* \Delta b_i$.

Since optimal value of the LP $\leq$ optimal value of the CP $\leq U$,
we have $\lambda_i^* \Delta b_i \leq U - v^*$, or

$$\Delta b_i \leq \frac{U - v^*}{\lambda_i^*}$$

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Since optimal value of the LP $\leq$ optimal value of the CP $\leq U$, we have $\lambda_i^* \Delta b_i \leq U - v^*$, or

$$\Delta b_i \leq \frac{U - v^*}{\lambda_i^*}$$

Since $\Delta b_i = A^i x - A^i x^* = A^i x - b_i$, this implies the inequality

$$A^i x \leq b_i + \frac{U - v^*}{\lambda_i^*}$$

…which can be propagated.

Example

$$\min\ 4x_1 + 7x_2$$
$$2x_1 + 3x_2 \ge 6 \qquad (\lambda_1 = 2)$$
$$2x_1 + x_2 \ge 4 \qquad (\lambda_1 = 0)$$
$$x_1, x_2 \ge 0$$

Suppose we have a feasible solution of the original CP with value U = 13.

Since the first constraint is tight, we can propagate the inequality

$$A^1 x \le b_1 + \frac{U - v^*}{\lambda_1^*}$$

or $$2x_1 + 3x_2 \le 6 + \frac{13 - 12}{2} = 6.5$$

Reduced-cost domain filtering

Suppose $x_j^* = 0$, which means the constraint $x_j \geq 0$ is tight.

The inequality $A^i x \leq b_i + \dfrac{U - v^*}{\lambda_i^*}$ becomes $x_j \leq \dfrac{U - v^*}{\boxed{r_j}}$

The dual multiplier for $x_j \geq 0$ is the reduced cost r_j of x_j, because increasing x_j (currently 0) by 1 increases optimal cost by r_j.

Similar reasoning can bound a variable below when it is at its upper bound.

Example

$$\min\ 4x_1 + 7x_2$$
$$2x_1 + 3x_2 \geq 6 \qquad (\lambda_1 = 2)$$
$$2x_1 + x_2 \geq 4 \qquad (\lambda_1 = 0)$$
$$x_1, x_2 \geq 0$$

Suppose we have a feasible solution of the original CP with value U = 13.

Since x_2* = 0, we have $x_2 \leq \dfrac{U - v^*}{r_2}$

or $x_2 \leq \dfrac{13-12}{2} = 0.5$

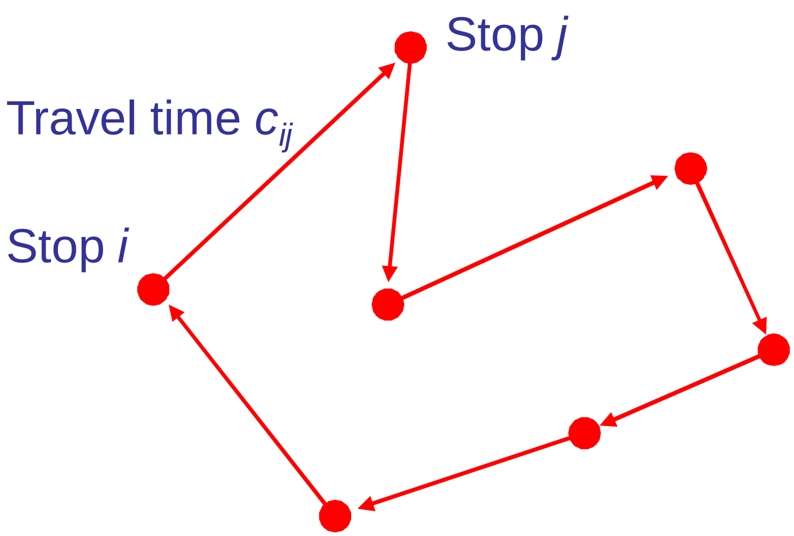
If x_2 is required to be integer, we can fix it to zero. This is **reduced-cost variable fixing.**

Example: Single-Vehicle Routing

A vehicle must make several stops and return home, perhaps subject to time windows.

The objective is to find the order of stops that minimizes travel time.

This is also known as the **traveling salesman problem** (with time windows).



Assignment Relaxation

$$\min \sum_{ij} c_{ij} x_{ij}$$

= 1 if stop *i* immediately precedes stop *j*

$$\sum_j x_{ij} = \sum_j x_{ji} = 1, \text{ all } i$$

Stop *i* is preceded and followed by exactly one stop.

$$x_{ij} \in \{0,1\}, \text{ all } i, j$$

Assignment Relaxation

$$\min \sum_{ij} c_{ij} x_{ij}$$

= 1 if stop *i* immediately precedes stop *j*

$$\sum_j x_{ij} = \sum_j x_{ji} = 1, \text{ all } i$$

Stop *i* is preceded and followed by exactly one stop.

$$0 \le x_{ij} \le 1, \text{ all } i, j$$

Because this problem is **totally unimodular**, it can be solved as an LP.

The relaxation provides a very weak lower bound on the optimal value.

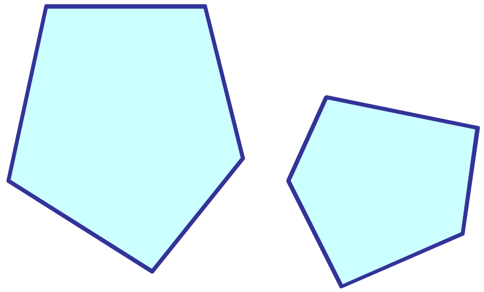
But **reduced-cost variable fixing** can be very useful in a CP context.

Disjunctions of linear systems

Disjunctions of linear systems often occur naturally in problems and can be given a convex hull relaxation.

A disjunction of linear systems represents a union of polyhedra.

$$\min \; cx$$
$$\bigvee_k \left(A^k x \geq b^k \right)$$



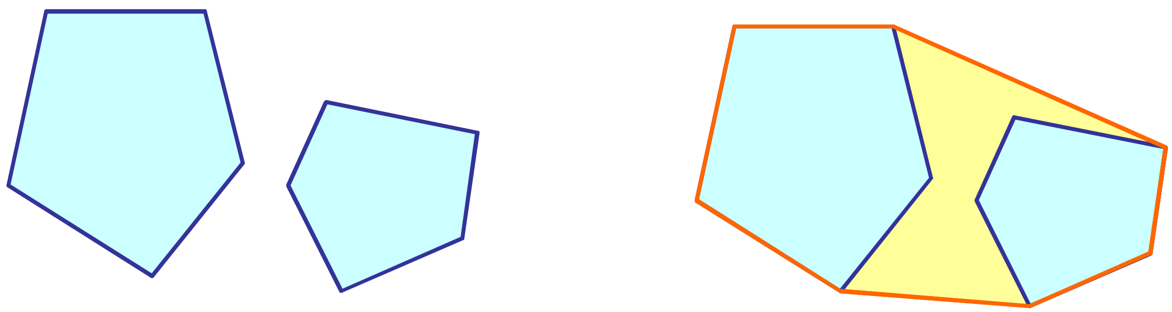
Relaxing a disjunction of linear systems

Disjunctions of linear systems often occur naturally in problems and can be given a convex hull relaxation.

A disjunction of linear systems represents a union of polyhedra.

We want a convex hull relaxation (tightest linear relaxation).

$$\min \; cx$$
$$\bigvee_k \left(A^k x \geq b^k \right)$$



Relaxing a disjunction of linear systems

Disjunctions of linear systems often occur naturally in problems and can be given a convex hull relaxation.

The closure of the convex hull of

$$
\begin{aligned}
&\min \; cx \\
&\bigvee_k \left(A^k x \geq b^k \right)
\end{aligned}
$$

…is described by

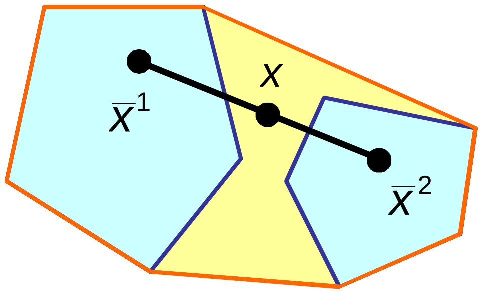
$$
\begin{aligned}
&\min \; cx \\
&A^k x^k \geq b^k y_k, \text{ all } k \\
&\sum_k y_k = 1 \\
&x = \sum_k x^k \\
&0 \leq y_k \leq 1
\end{aligned}
$$

Why?

To derive convex hull relaxation of a disjunction…

Write each solution as a convex combination of points in the polyhedron

$$
\begin{aligned}
&\min \ cx \\
&A^k \bar{x}^k \geq b^k, \ \text{all } k \\
&\sum_k y_k = 1 \\
&x = \sum_k y_k \bar{x}^k \\
&0 \leq y_k \leq 1
\end{aligned}
$$



Convex hull relaxation (tightest linear relaxation)

Why?

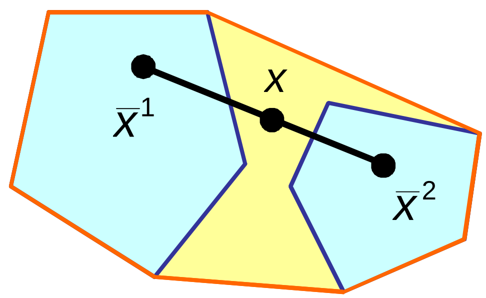
To derive convex hull relaxation of a disjunction…

Write each solution as a convex combination of points in the polyhedron

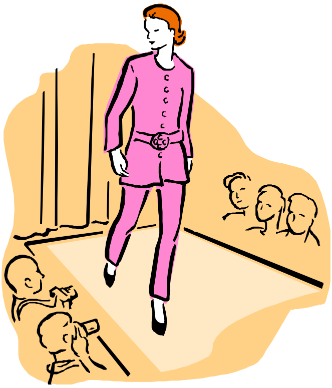
$$\begin{aligned} &\min\ cx \\ &A^k \bar{x}^k \geq b^k, \text{ all } k \\ &\sum_k y_k = 1 \\ &x = \sum_k y_k \bar{x}^k \\ &0 \leq y_k \leq 1 \end{aligned}$$

Change of variable $x = y_k \bar{x}^k$

$$\begin{aligned} &\min\ cx \\ &A^k x^k \geq b^k y_k, \text{ all } k \\ &\sum_k y_k = 1 \\ &x = \sum_k x^k \\ &0 \leq y_k \leq 1 \end{aligned}$$



Convex hull relaxation
(tightest linear relaxation)



Mixed Integer/Linear Modeling

MILP Representability

Disjunctive Modeling

Knapsack Modeling

Motivation

A **mixed integer/linear programming** (MILP) problem has the form

$$\begin{aligned} &\min \; cx + dy \\ &Ax + By \geq b \\ &x, y \geq 0 \\ &y \text{ integer} \end{aligned}$$

- We can **relax** a CP problem by modeling some constraints with an MILP.
- If desired, we can then **relax the MILP** by dropping the integrality constraint, to obtain an LP.
- The LP relaxation can be strengthened with **cutting planes**.
- The first step is to learn **how to write** MILP models.

MILP Representability

A subset S of $\mathbb{R}^n$ is **MILP representable** if it is the projection onto x of some MILP constraint set of the form

$$Ax + Bu + Dy \geq b$$

$$x, y \geq 0$$

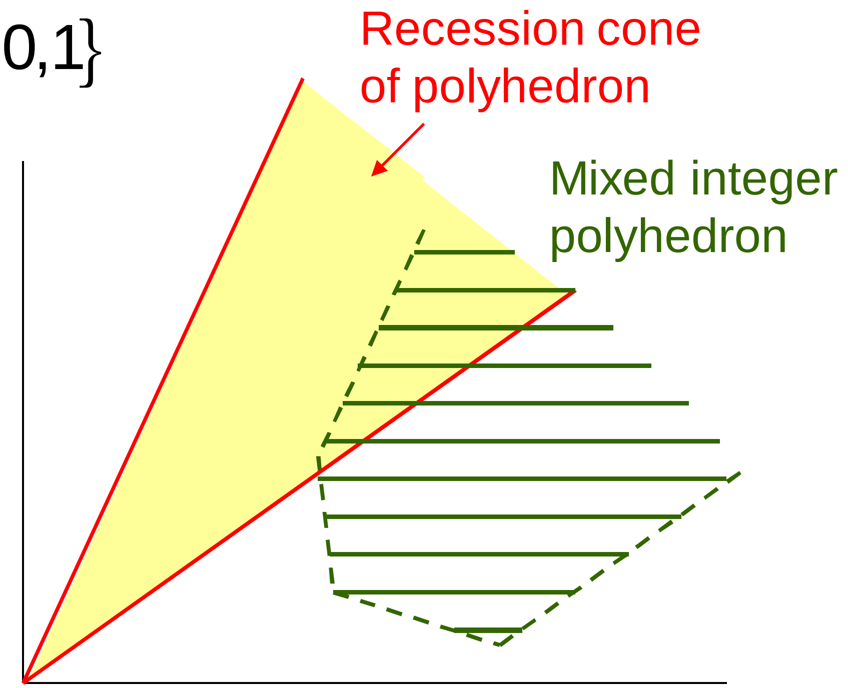
$$x \in \mathbb{R}^n \times \mathbb{Z}^p,\ \ u \in \mathbb{R}^m,\ \ y_k \in \{0,1\}$$

MILP Representability

A subset S of $\mathbb{R}^n$ is **MILP representable** if it is the projection onto x of some MILP constraint set of the form

$$Ax + Bu + Dy \geq b$$
$$x, y \geq 0$$
$$x \in \mathbb{R}^n \times \mathbb{Z}^p,\ \ u \in \mathbb{R}^m,\ \ y_k \in \{0,1\}$$

Theorem. $S \subset \mathbb{R}^n$ is MILP representable if and only if S is the union of finitely many *mixed integer polyhedra* having the same *recession cone*.



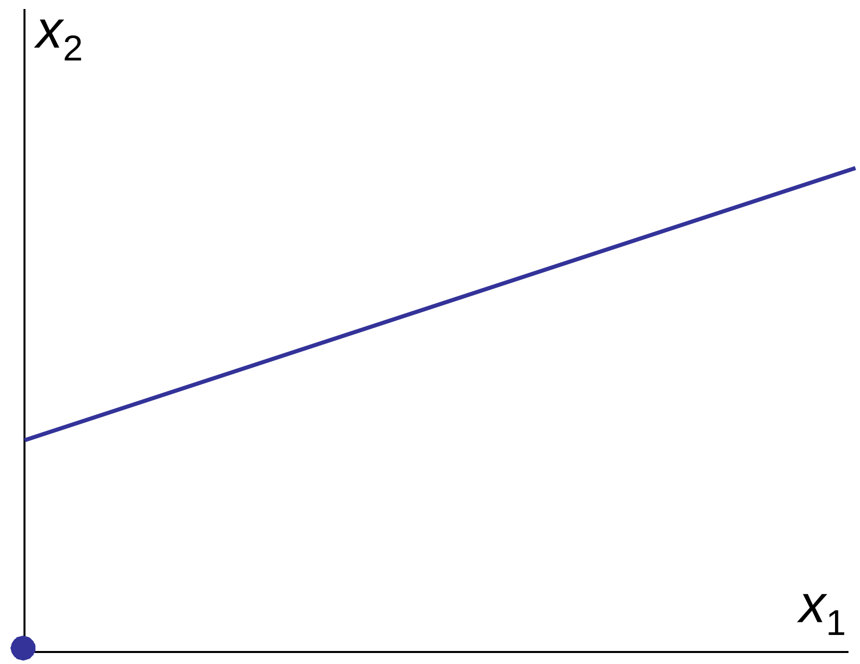
Example: Fixed charge function

Minimize a fixed charge function:

$$\min\ x_2$$

$$x_2 \geq \begin{Bmatrix} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{Bmatrix}$$

$$x_1 \geq 0$$



Example

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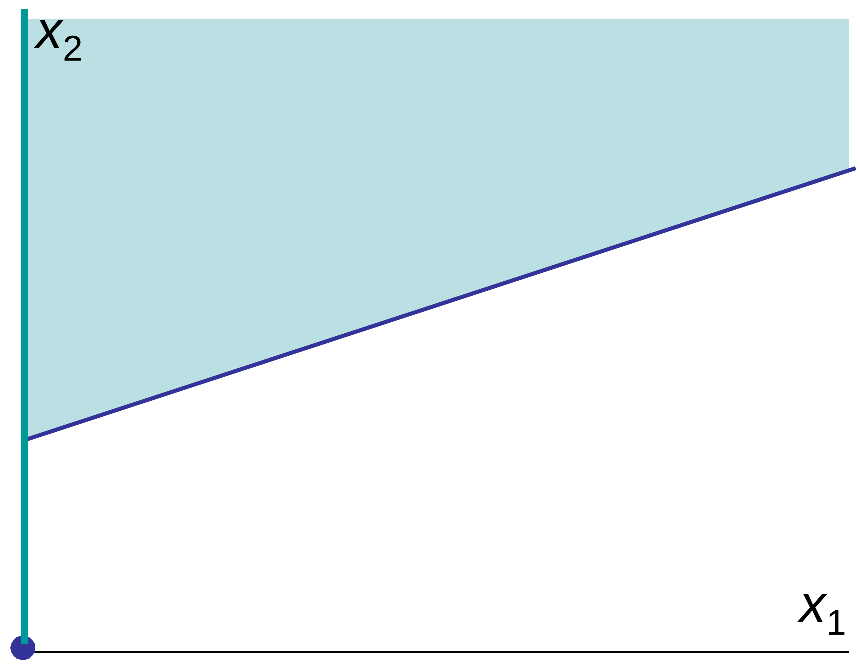
Feasible set



Example

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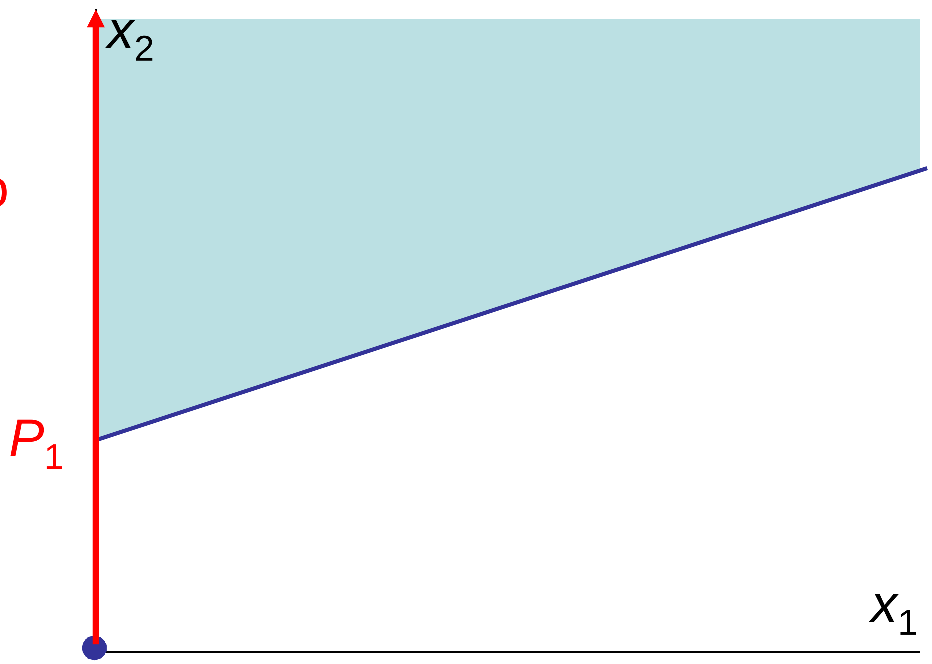
$$x_2 \geq \begin{Bmatrix} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{Bmatrix}$$

$$x_1 \geq 0$$



Union of two polyhedra P_1, P_2

Example

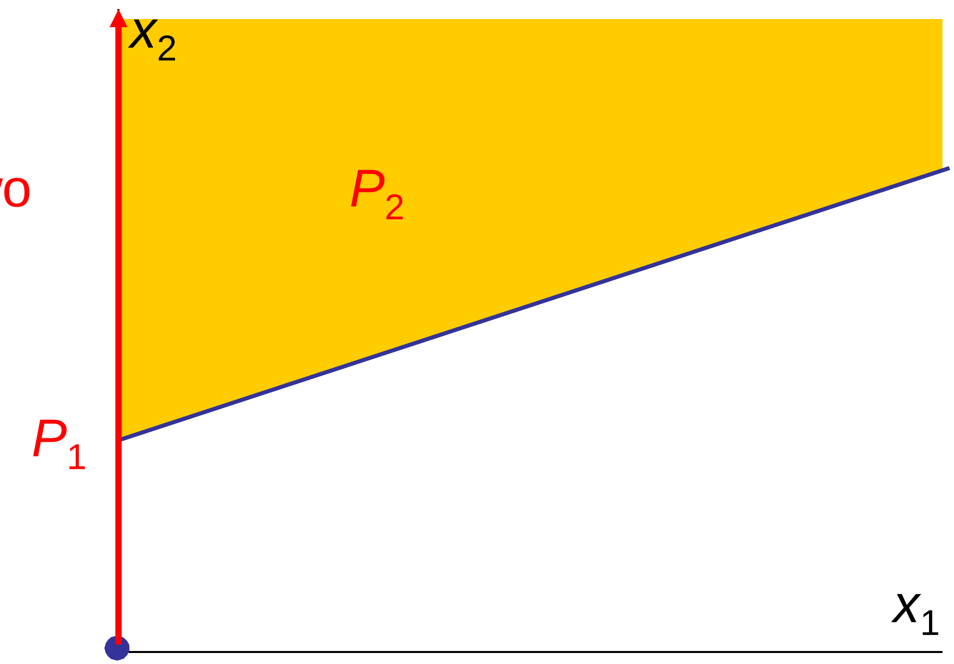
Minimize a fixed charge function:

$$\min \ x_2$$

$$x_2 \geq \begin{Bmatrix} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{Bmatrix}$$

$$x_1 \geq 0$$

Union of two polyhedra
P_1, P_2



Example

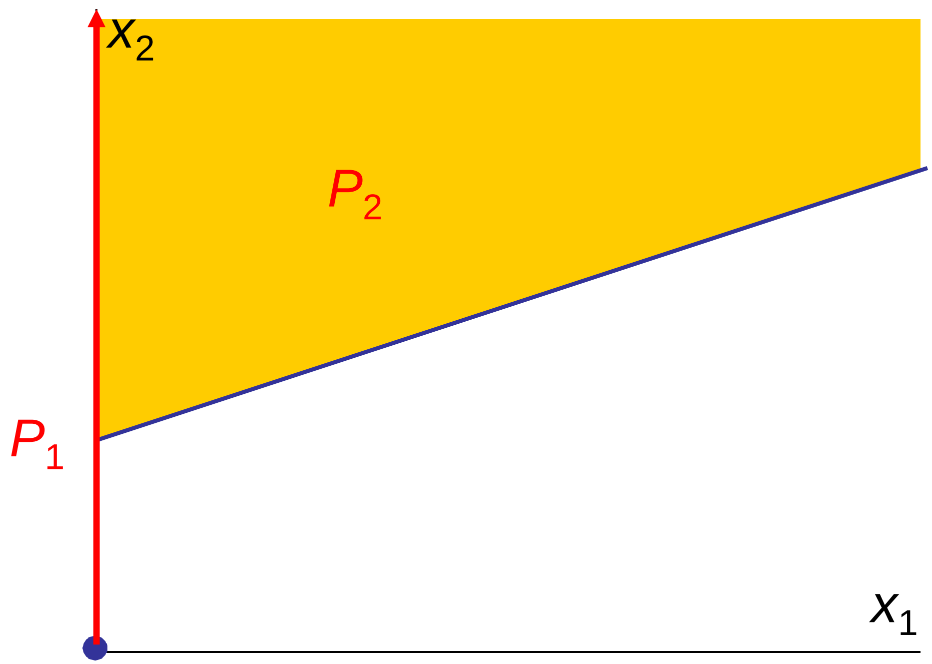
Minimize a fixed charge function:

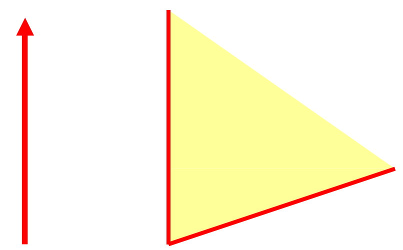
$$\min \ x_2$$

$$x_2 \geq \begin{Bmatrix} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{Bmatrix}$$

$$x_1 \geq 0$$

The polyhedra have different recession cones.





P_1 recession cone

P_2 recession cone

Example

Minimize a fixed charge function:

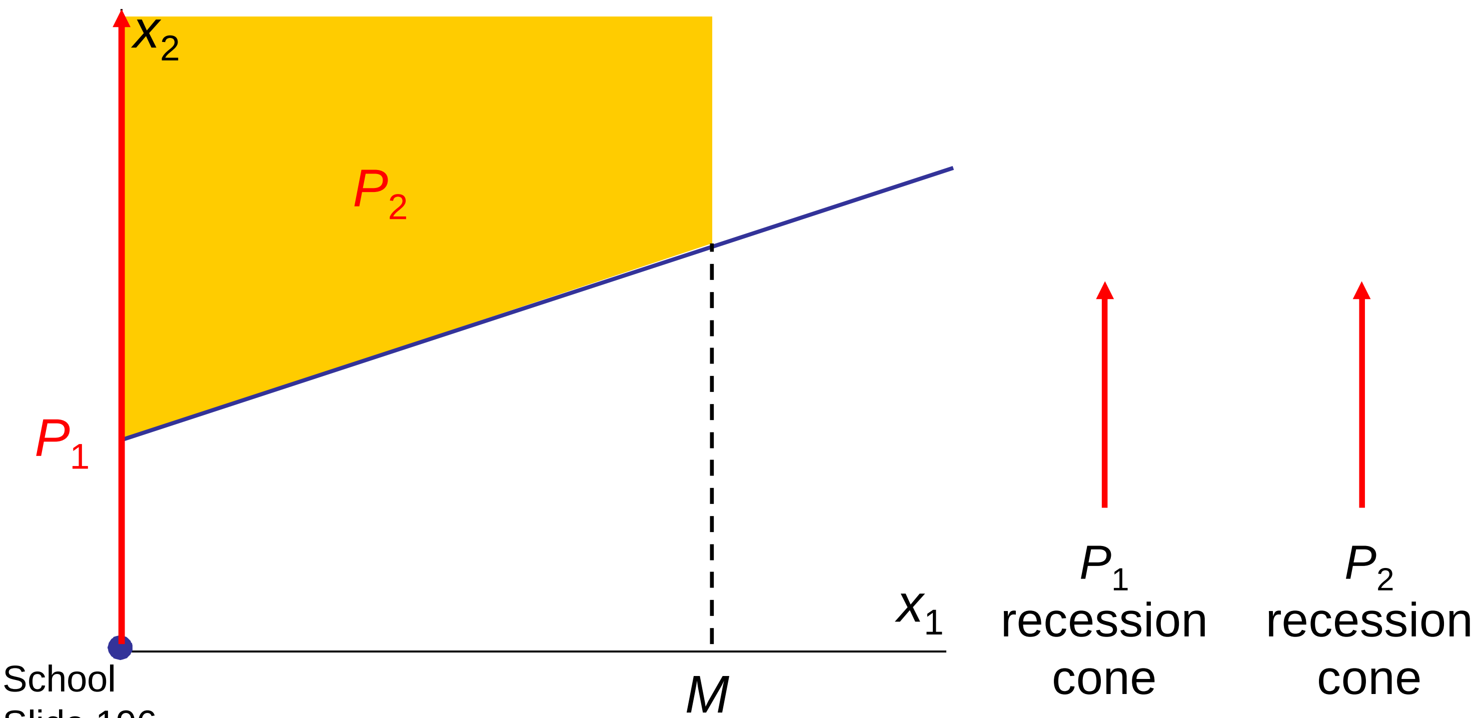
Add an upper bound on x_1

$$\min \; x_2$$

$$x_2 \geq \begin{Bmatrix} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{Bmatrix}$$

$$0 \leq x_1 \leq M$$

The polyhedra have the same recession cone.



Modeling a union of polyhedra

Start with a disjunction of linear systems to represent the union of polyhedra.

The *k*th polyhedron is $\{x \mid A^k x \geq b\}$

$$\min \; cx$$
$$\bigvee_k \left(A^k x \geq b^k \right)$$

Introduce a 0-1 variable y_k that is 1 when x is in polyhedron $\underline{k}$.

Disaggregate x to create an x^k for each k.

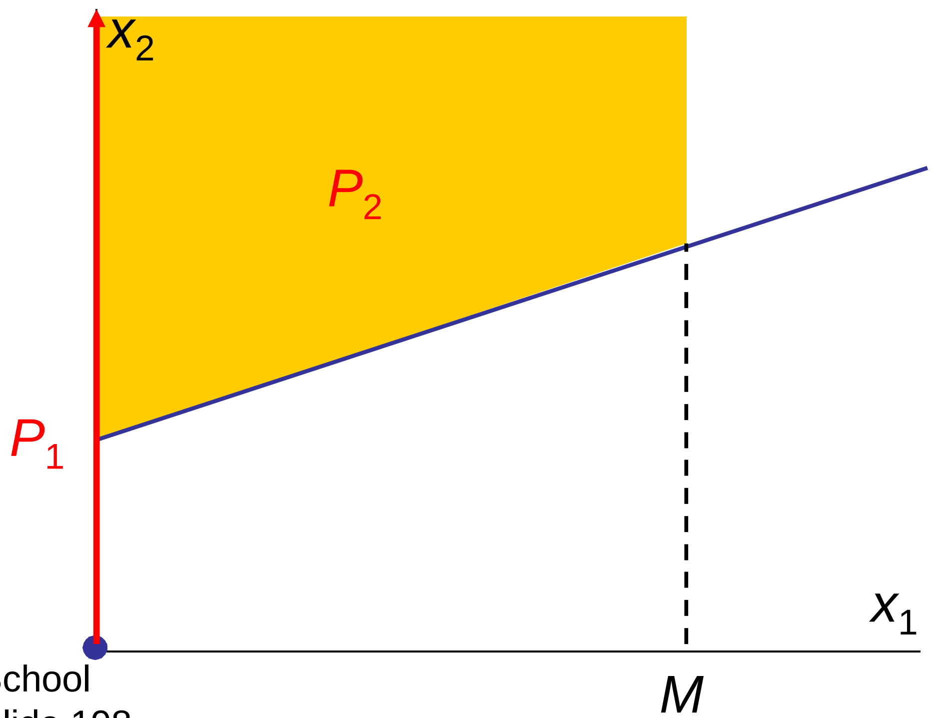
$$\min \; cx$$
$$A^k x^k \geq b^k y_k, \text{ all } k$$
$$\sum_k y_k = 1$$
$$x = \sum_k x^k$$
$$y_k \in \{0,1\}$$

Example

Start with a disjunction of linear systems to represent the union of polyhedra

$$\min \; x_2$$

$$\begin{pmatrix} x_1 = 0 \\ x_2 \ge 0 \end{pmatrix} \vee \begin{pmatrix} 0 \le x_1 \le M \\ x_2 \ge f + cx_1 \end{pmatrix}$$



Example

Start with a disjunction of linear systems to represent the union of polyhedra

$$\min\ x_2$$
$$\begin{pmatrix} x_1 = 0 \\ x_2 \ge 0 \end{pmatrix} \vee \begin{pmatrix} 0 \le x_1 \le M \\ x_2 \ge f + cx_1 \end{pmatrix}$$

Introduce a 0-1 variable y_k that is 1 when x is in polyhedron $\underline{k}$.

Disaggregate x to create an x^k for each k.

$$\min\ cx$$
$$x_1^1 = 0,\ \ x_2^1 \ge 0$$
$$0 \le x_1^2 \le My_2, \quad -cx_1^2 + x_2^2 \ge fy_2$$
$$y_1 + y_2 = 1,\ \ y_k \in \{0,1\}$$
$$x = x^1 + x^2$$

Example

To simplify:

Replace x_1^2 with x_1.

Replace x_2^2 with x_2.

Replace y_2 with y.

$$\min \ x_2$$
$$x_1^1 = 0, \ \ x_2^1 \geq 0$$
$$0 \leq x_1^2 \leq My_2, \quad -cx_1^2 + x_2^2 \geq fy_2$$
$$y_1 + y_2 = 1, \ \ y_k \in \{0,1\}$$
$$x = x^1 + x^2$$

This yields

$$\min \ x_2$$
$$0 \leq x_1 \leq My$$
$$x_2 \geq fy + cx_1$$
$$y \in \{0,1\}$$

or

$$\min \ fy + cx$$
$$0 \leq x \leq \boxed{M}y$$
$$y \in \{0,1\}$$

"Big *M* "

Disjunctive Modeling

Disjunctions often occur naturally in problems and can be given an MILP model.

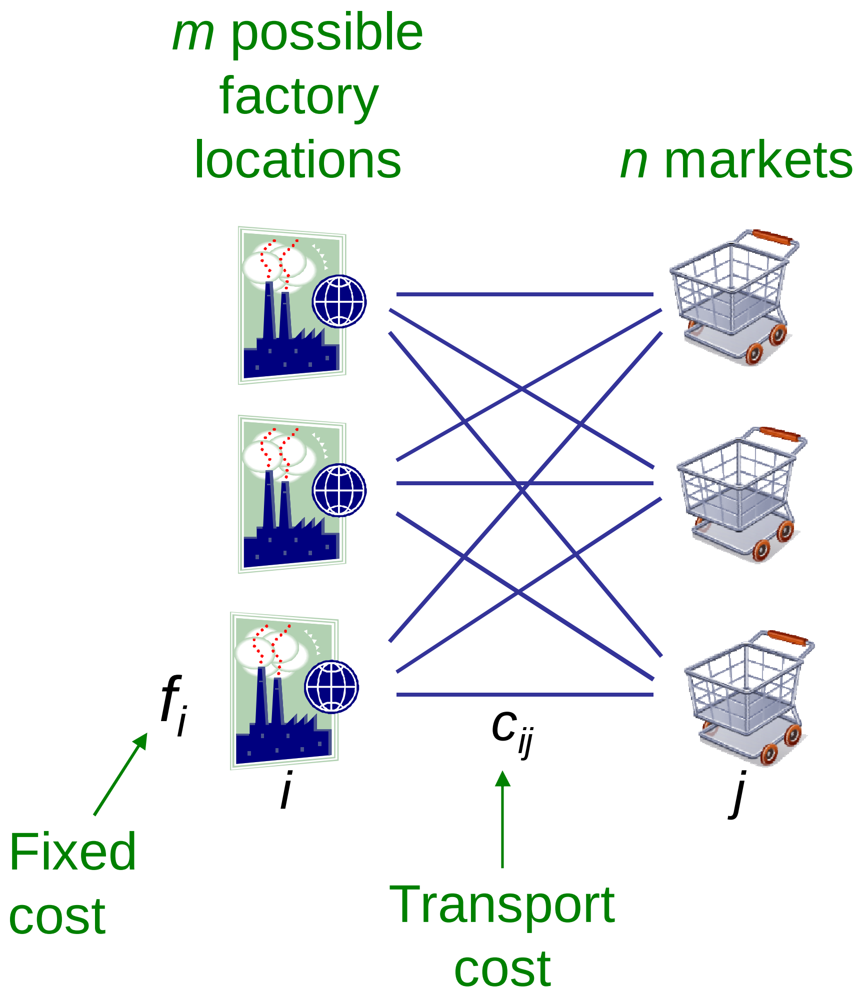
Recall that a disjunction of linear systems (representing polyhedra with the same recession cone)

$$\min \ cx$$
$$\bigvee_k \left(A^k x \geq b^k \right)$$

…has the MILP model

$$\begin{aligned} & \min \ cx \\ & A^k x^k \geq b^k y_k, \ \text{all } k \\ & \sum_k y_k = 1 \\ & x = \sum_k x^k \\ & y_k \in \{0,1\} \end{aligned}$$

Example: Uncapacitated facility location



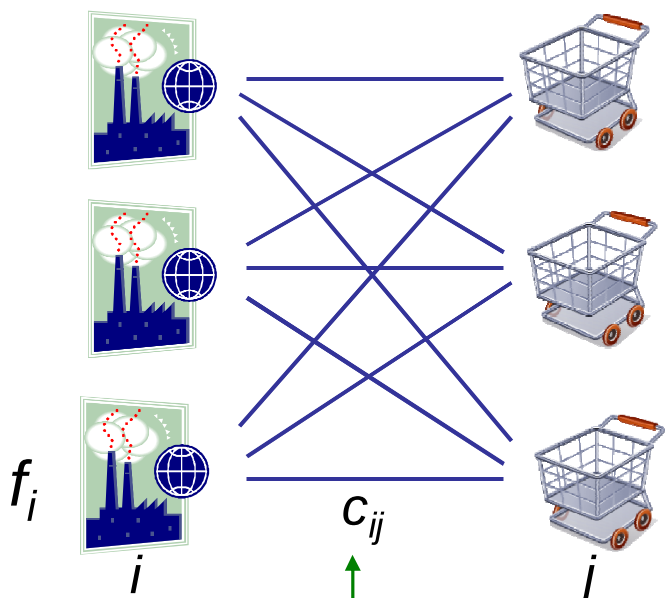
Locate factories to serve markets so as to minimize total fixed cost and transport cost.

No limit on production capacity of each factory.

Uncapacitated facility location

m possible factory locations

n markets



Fixed cost

Transport cost

Fraction of market *j*'s demand satisfied from location *i*

Disjunctive model:

$$\min \sum_i z_i + \sum_{ij} c_{ij} \boxed{x_{ij}}$$

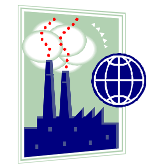
$$\begin{pmatrix} x_{ij} = 0, \ \text{all } j \\ z_i = 0 \end{pmatrix} \vee \begin{pmatrix} 0 \le x_{ij} \le 1, \ \text{all } j \\ z_i \ge f_i \end{pmatrix}, \quad \text{all } i$$

$$\sum_i x_{ij} = 1, \quad \text{all } j$$

No factory at location *i*

Factory at location *i*

Uncapacitated facility location



MILP formulation:

$$\min \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij}$$

$$0 \le x_{ij} \le y_i, \quad \text{all } i, j$$

$$y_i \in \{0,1\}$$

Disjunctive model:

$$\min \sum_i z_i + \sum_{ij} c_{ij} x_{ij}$$

$$\begin{pmatrix} x_{ij} = 0, \ \text{all } j \\ z_i = 0 \end{pmatrix} \vee \begin{pmatrix} 0 \le x_{ij} \le 1, \ \text{all } j \\ z_i \ge f_i \end{pmatrix}, \quad \text{all } i$$

$$\sum_i x_{ij} = 1, \quad \text{all } j$$

No factory at location *i*

Factory at location *i*

Uncapacitated facility location

Maximum output
from location i

MILP formulation:

$$\min \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij}$$

$$0 \le x_{ij} \le y_i, \quad \text{all } i, j$$

$$y_i \in \{0,1\}$$

Beginner's model:

$$\min \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij}$$

$$\sum_j x_{ij} \le \boxed{n y_i}, \quad \text{all } i, j$$

$$y_i \in \{0,1\}$$

Based on capacitated location model.

It has a **weaker continuous relaxation**
(obtained by replacing $y_i \in \{0,1\}$ with $0 \le y_i \le 1$).

This beginner's mistake can be avoided by
starting with disjunctive formulation.

Knapsack Modeling

- Knapsack models consist of **knapsack covering** and **knapsack packing** constraints.
- The freight transfer model presented earlier is an example.
- We will consider a similar example that combines disjunctive and knapsack modeling.
- Most OR professionals are unlikely to write a model as good as the one presented here.

Note on tightness of knapsack models

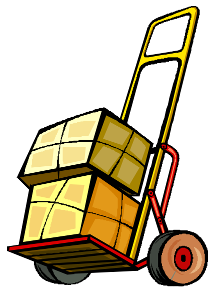
- The continuous relaxation of a knapsack model is not in general a convex hull relaxation.

    - A disjunctive formulation would provide a convex hull relaxation, but there are exponentially many disjuncts.

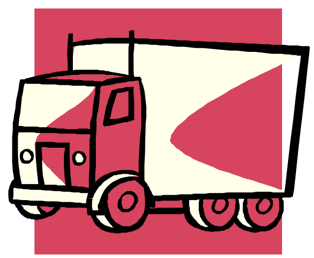
- Knapsack cuts can significantly tighten the relaxation.

Example: Package transport

Each package j has size a_j



Each truck i has capacity Q_i and costs c_i to operate



Disjunctive model

Knapsack constraints

$$\min \sum_i z_i$$

$$\sum_i Q_i y_i \ge \sum_j a_j; \quad \sum_i x_{ij} = 1, \text{ all } j$$

$$\begin{pmatrix} y_i = 1 \\ z_i = c_i \\ \sum_j a_j x_{ij} \le Q_i \\ \boxed{x_{ij}} \in \{0,1\}, \text{ all } j \end{pmatrix} \vee \begin{pmatrix} y_i = 0 \\ z_i = 0 \\ x_{ij} = 0 \end{pmatrix}, \quad \text{all } i$$

$$\boxed{y_i} \in \{0,1\}$$

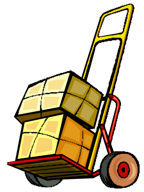
Truck i used

Truck i not used

1 if truck i carries package j

1 if truck i is used

Example: Package transport



MILP model

$$\min \sum_i c_i y_i$$

$$\sum_i Q_i y_i \geq \sum_j a_j; \quad \sum_i x_{ij} = 1, \text{ all } j$$

$$\sum_j a_j x_{ij} \leq Q_i y_i, \quad \text{all } i$$

$$x_{ij} \leq y_i, \quad \text{all } i, j$$

$$x_{ij}, y_i \in \{0,1\}$$

Disjunctive model

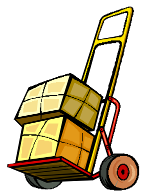
$$\min \sum_i z_i$$

$$\sum_i Q_i y_i \geq \sum_j a_j; \quad \sum_i x_{ij} = 1, \text{ all } j$$

$$\begin{pmatrix} y_i = 1 \\ z_i = c_i \\ \sum_j a_j x_{ij} \leq Q_i \\ x_{ij} \in \{0,1\}, \text{ all } j \end{pmatrix} \vee \begin{pmatrix} y_i = 0 \\ z_i = 0 \\ x_{ij} = 0 \end{pmatrix}, \quad \text{all } i$$

$$y_i \in \{0,1\}$$

Example: Package transport



MILP model

$$\min \sum_i c_i y_i$$

$$\boxed{\sum_i Q_i y_i \geq \sum_j a_j}; \quad \sum_i x_{ij} = 1, \text{ all } j$$

$$\sum_j a_j x_{ij} \leq Q_i \boxed{y_i}, \quad \text{all } i$$

$$x_{ij} \leq y_i, \quad \text{all } i, j$$

$$x_{ij}, y_i \in \{0,1\}$$

Most OR professionals would omit this constraint, since it is the sum over *i* of the next constraint. But it generates very effective knapsack cuts.

Modeling trick; unobvious without disjunctive approach

Network Flows and Filtering

Min Cost Network Flow

Max Flow

Filtering: Cardinality

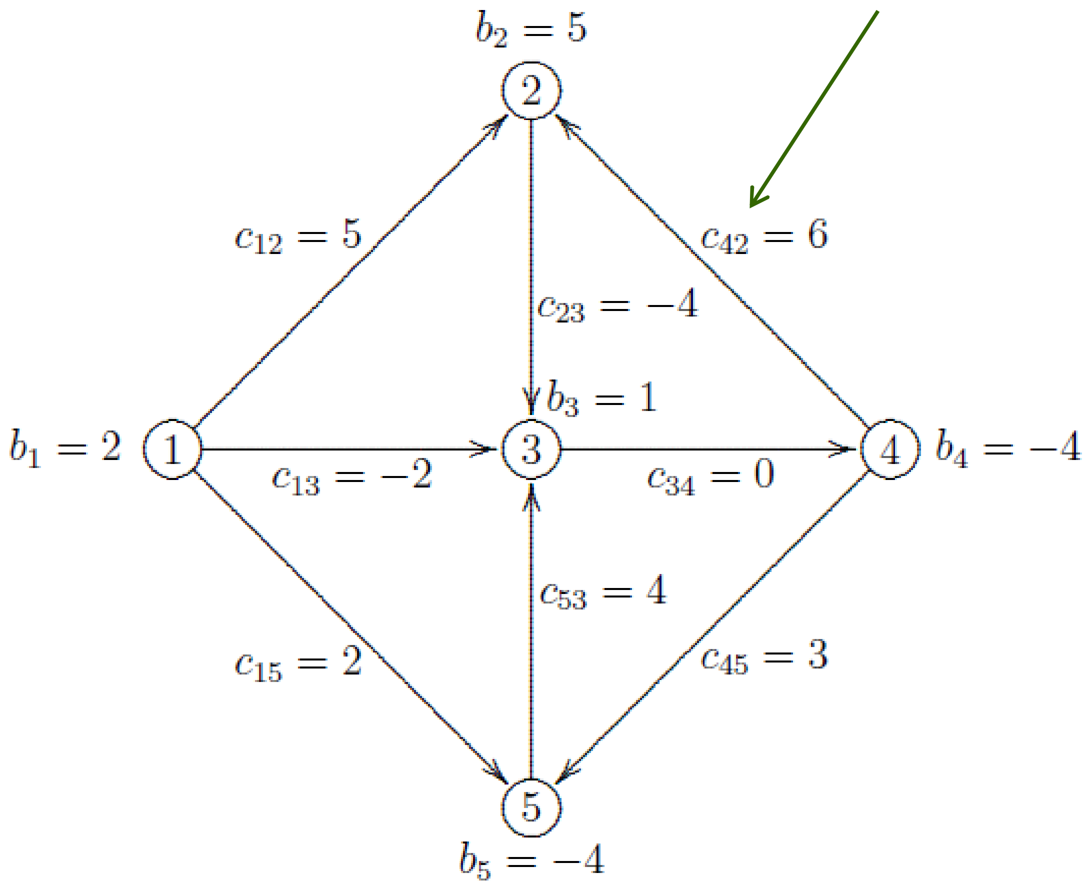
Filtering: Sequence

Min Cost Network Flow

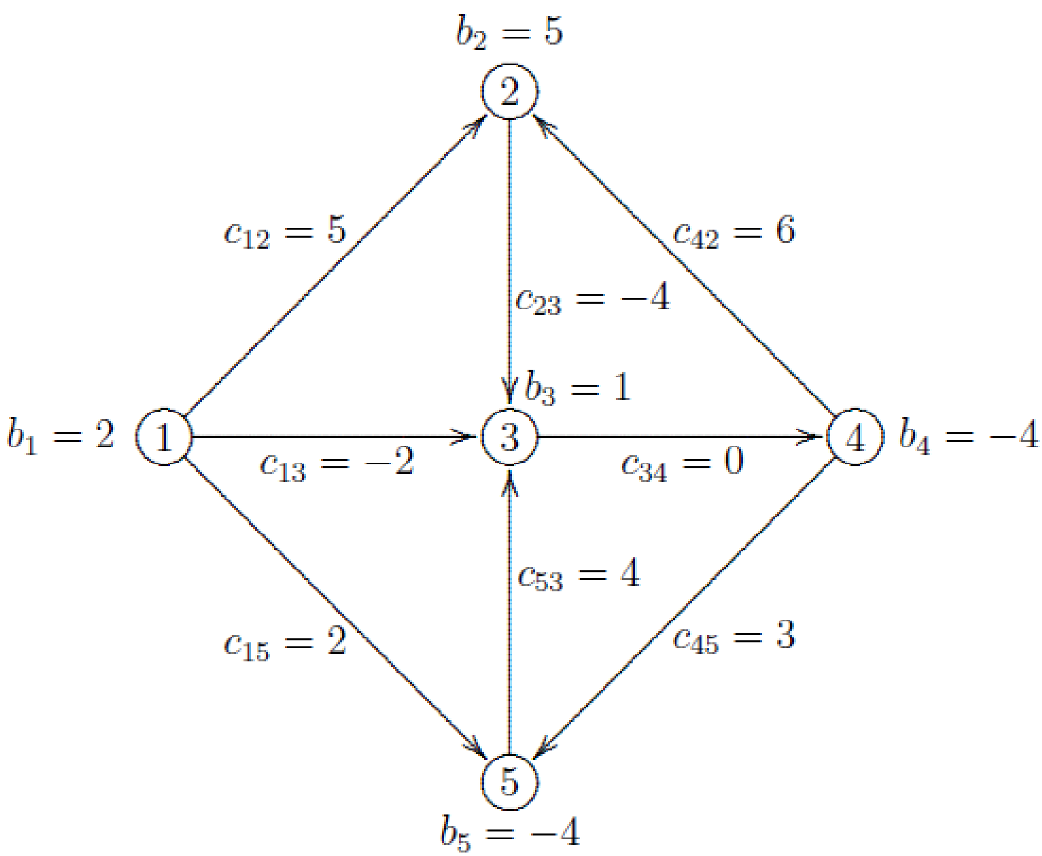
A min cost network flow problem:

Net supply at node

Unit cost of flow



Min Cost Network Flow



Flow on arc (i,j)

In general:

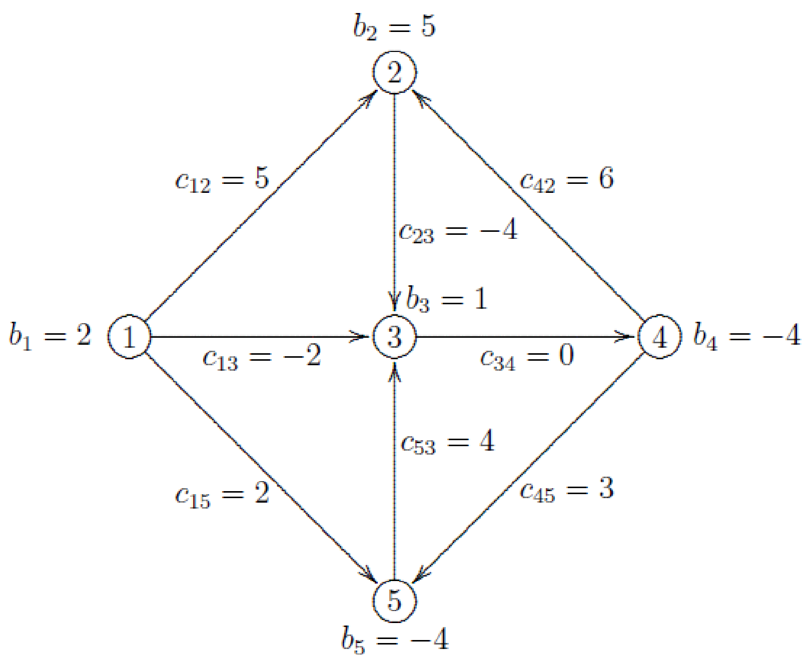
$$\min \sum_{ij} c_{ij} x_{ij}$$

$$\sum_j x_{ij} - \sum_j x_{ji} = b_i, \ \text{all } i$$

$$x_{ij} \geq 0, \ \text{all } i, j$$

This is an LP.

Min Cost Network Flow



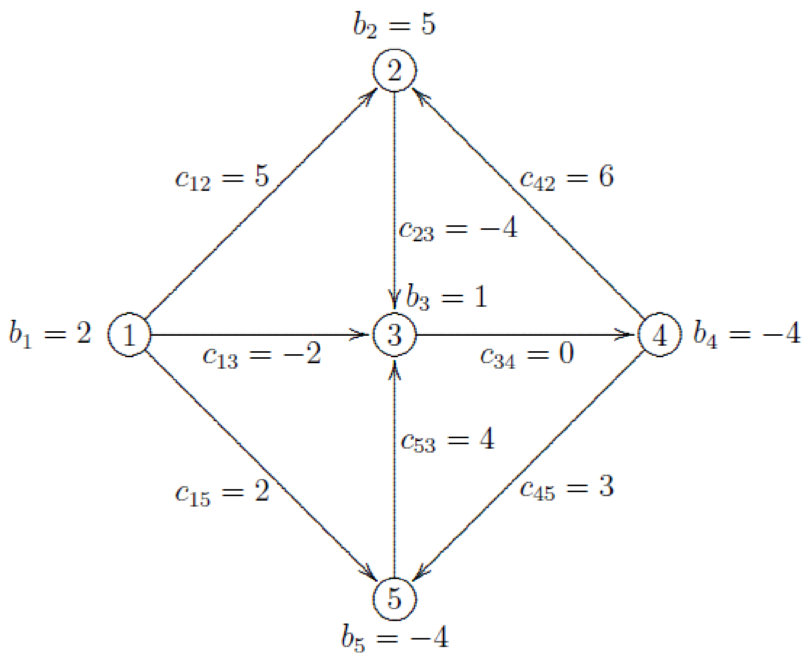
Matrix form:

$$\min \sum_{ij} c_{ij} x_{ij}$$

$$\begin{bmatrix} 1 & 1 & 1 & & & & & \\ -1 & & & 1 & & -1 & & \\ & -1 & & -1 & 1 & & & -1 \\ & & & & -1 & 1 & 1 & \\ & & -1 & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{13} \\ x_{15} \\ x_{23} \\ x_{34} \\ x_{42} \\ x_{45} \\ x_{53} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \\ -4 \\ -4 \end{bmatrix}$$

$$x_{ij} \geq 0, \text{ all } i, j$$

Min Cost Network Flow



Matrix form:

$$\min \sum_{ij} c_{ij} x_{ij}$$

$$\begin{bmatrix} 1 & 1 & 1 & & & & & \\ -1 & & & 1 & & -1 & & \\ & -1 & & -1 & 1 & & & -1 \\ & & & & -1 & 1 & 1 & \\ & & -1 & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{13} \\ x_{15} \\ x_{23} \\ x_{34} \\ x_{42} \\ x_{45} \\ x_{53} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \\ -4 \\ -4 \end{bmatrix}$$

$$x_{ij} \geq 0, \text{ all } i, j$$

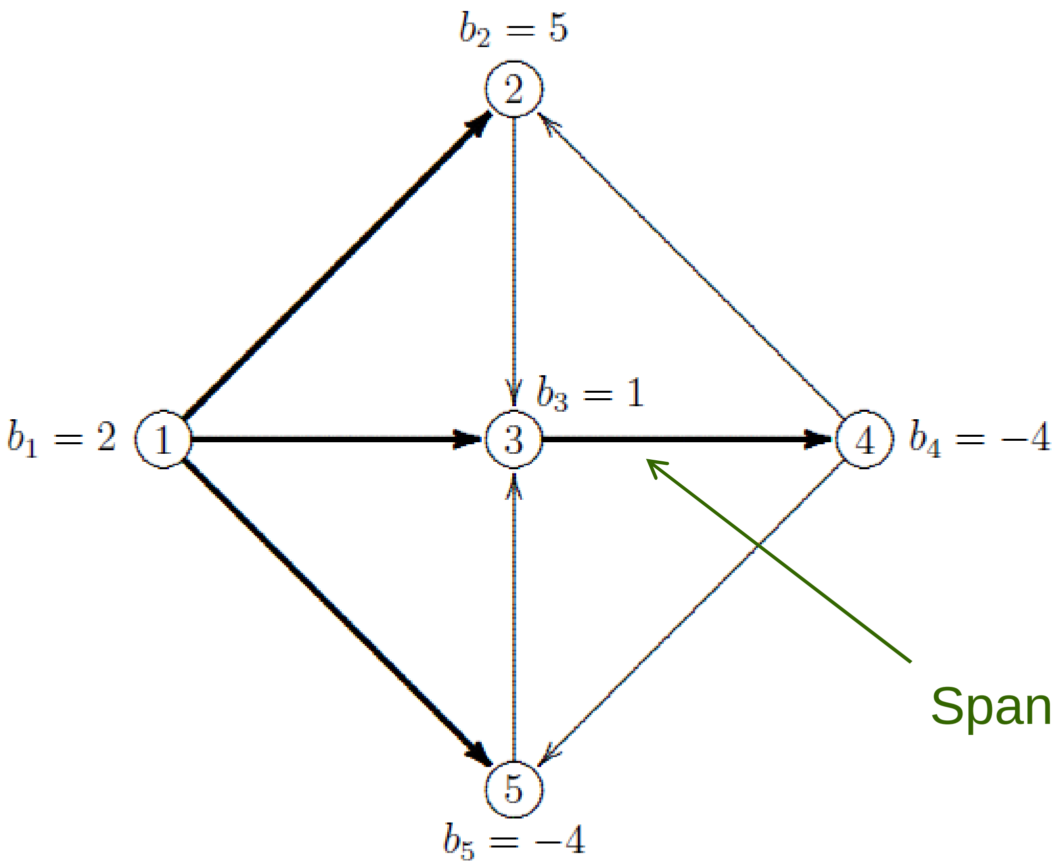
Rows sum to zero.

So rank < *m* (= # of nodes)

Will show rank = *m* – 1

Min Cost Network Flow

Basis tree theorem. Every basis corresponds to a spanning tree.



Spanning tree

$$
\begin{array}{cc}
\begin{array}{cccc} x_{12} & x_{13} & x_{15} & x_{34} \end{array} & \\
\begin{bmatrix} 1 & 1 & 1 & \\ -1 & & & \\ & -1 & & 1 \\ & & & -1 \\ & & -1 & \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\end{array}
$$

Corresponding columns

Min Cost Network Flow

$$\begin{array}{cc} \begin{array}{cccc} x_{12} & x_{13} & x_{15} & x_{34} \end{array} & \\ \left[\begin{array}{cccc} 1 & 1 & 1 & \\ -1 & & & \\ & -1 & & 1 \\ & & & -1 \\ & & -1 & \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \end{array}$$

Can triangularize (except for last row) by permuting rows, columns:

(a)

$$\begin{array}{cc} \begin{array}{cccc} x_{12} & x_{13} & x_{15} & x_{34} \end{array} & \\ \left[\begin{array}{cccc} & & & -1 \\ 1 & 1 & 1 & \\ -1 & & & \\ & -1 & & 1 \\ & & -1 & \end{array}\right] & \begin{array}{c} 4 \\ 1 \\ 2 \\ 3 \\ 5 \end{array} \end{array}$$

(b)

$$\begin{array}{cc} \begin{array}{cccc} x_{34} & x_{12} & x_{13} & x_{15} \end{array} & \\ \left[\begin{array}{cccc} -1 & & & \\ & 1 & 1 & 1 \\ & -1 & & \\ 1 & & -1 & \\ & & & -1 \end{array}\right] & \begin{array}{c} 4 \\ 1 \\ 2 \\ 3 \\ 5 \end{array} \end{array}$$

(c)

$$\begin{array}{cc} \begin{array}{cccc} x_{34} & x_{13} & x_{12} & x_{15} \end{array} & \\ \left[\begin{array}{cccc} -1 & & & \\ 1 & -1 & & \\ & 1 & 1 & 1 \\ & & -1 & \\ & & & -1 \end{array}\right] & \begin{array}{c} 4 \\ 3 \\ 1 \\ 2 \\ 5 \end{array} \end{array}$$

(d)

$$\begin{array}{cc} \begin{array}{cccc} x_{34} & x_{13} & x_{12} & x_{15} \end{array} & \\ \left[\begin{array}{cccc} -1 & & & \\ & -1 & & \\ & & -1 & \\ & 1 & 1 & 1 \\ & & & -1 \end{array}\right] & \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \\ 5 \end{array} \end{array}$$

Min Cost Network Flow

$$
\begin{array}{cc}
\begin{array}{cccc} x_{12} & x_{13} & x_{15} & x_{34} \end{array} & \\
\left[\begin{array}{cccc} 1 & 1 & 1 & \\ -1 & & & \\ & -1 & & 1 \\ & & & -1 \\ & & -1 & \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\end{array}
$$

Can triangularize (except for last row) by permuting rows, columns:

(a)

$$
\begin{array}{cc}
\begin{array}{cccc} x_{12} & x_{13} & x_{15} & x_{34} \end{array} & \\
\left[\begin{array}{cccc} & & & -1 \\ 1 & 1 & 1 & \\ -1 & & & \\ & -1 & & 1 \\ & & -1 & \end{array}\right] & \begin{array}{c} 4 \\ 1 \\ 2 \\ 3 \\ 5 \end{array}
\end{array}
$$

(b)

$$
\begin{array}{cc}
\begin{array}{cccc} x_{34} & x_{12} & x_{13} & x_{15} \end{array} & \\
\left[\begin{array}{cccc} -1 & & & \\ & 1 & 1 & 1 \\ & -1 & & \\ 1 & & -1 & \\ & & & -1 \end{array}\right] & \begin{array}{c} 4 \\ 1 \\ 2 \\ 3 \\ 5 \end{array}
\end{array}
$$

(c)

$$
\begin{array}{cc}
\begin{array}{cccc} x_{34} & x_{13} & x_{12} & x_{15} \end{array} & \\
\left[\begin{array}{cccc} -1 & & & \\ 1 & -1 & & \\ & 1 & 1 & 1 \\ & & -1 & \\ & & & -1 \end{array}\right] & \begin{array}{c} 4 \\ 3 \\ 1 \\ 2 \\ 5 \end{array}
\end{array}
$$

(d)

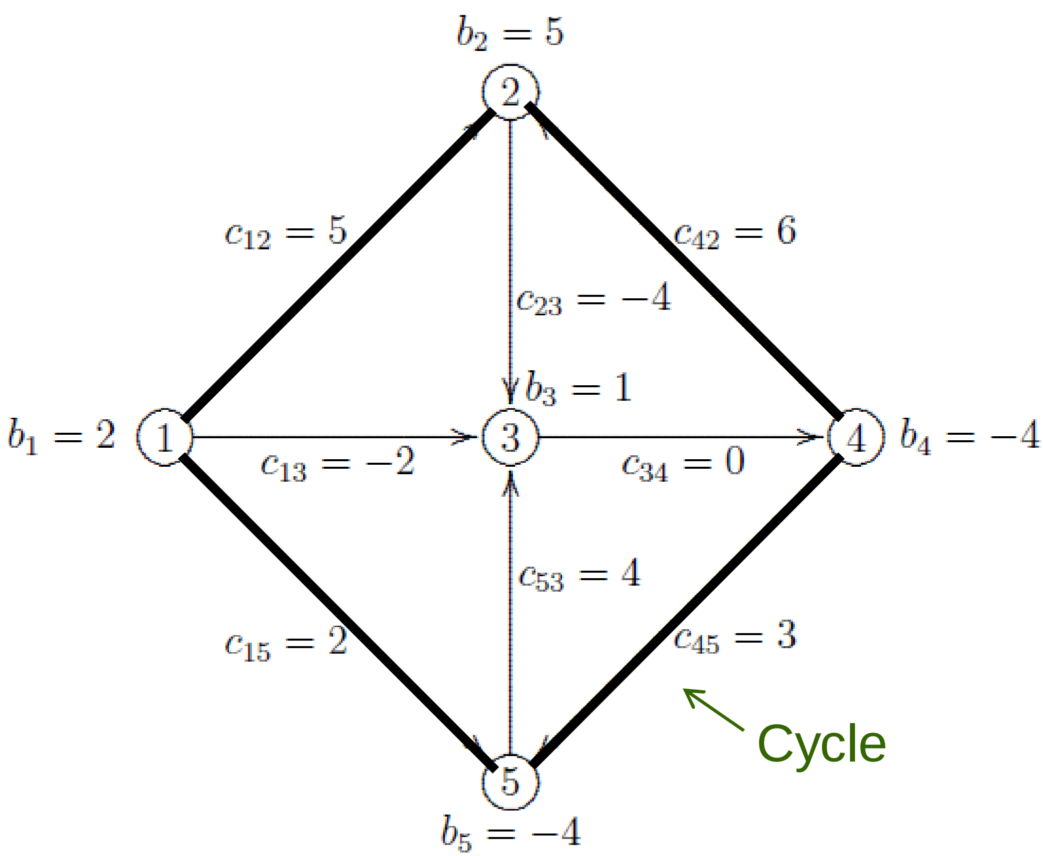
$$
\begin{array}{cc}
\begin{array}{cccc} x_{34} & x_{13} & x_{12} & x_{15} \end{array} & \\
\left[\begin{array}{cccc} -1 & & & \\ & -1 & & \\ & & -1 & \\ & 1 & 1 & 1 \\ & & & -1 \end{array}\right] & \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \\ 5 \end{array}
\end{array}
$$

So columns have rank m – 1 and form a basis.

Min Cost Network Flow

Conversely, any basis corresponds to a spanning tree.

Why? Columns corresponding to a cycle are linearly dependent and therefore not part of a basis.



Linearly dependent columns:

$$
\begin{array}{cc}
\begin{array}{cccc} x_{12} & x_{42} & x_{45} & x_{15} \end{array} & \\
\begin{bmatrix} 1 & & & 1 \\ -1 & -1 & & \\ & & & \\ & 1 & 1 & \\ & & -1 & -1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \\
\begin{array}{cccc} 1 & -1 & 1 & -1 \end{array} &
\end{array}
$$

Multiplier for backward arc

Multiplier for forward arc

Optimality conditions

Recall that basic solution $x_B = B^{-1}b$ is optimal if reduced cost vector $r = c_N - uN \geq 0$, where $u = c_B B^{-1}$

But $N_{ij} = e_i - e_j$, which means

$$r_{ij} = c_{ij} - u(e_i - e_j) = c_{ij} - u_i + u_j$$

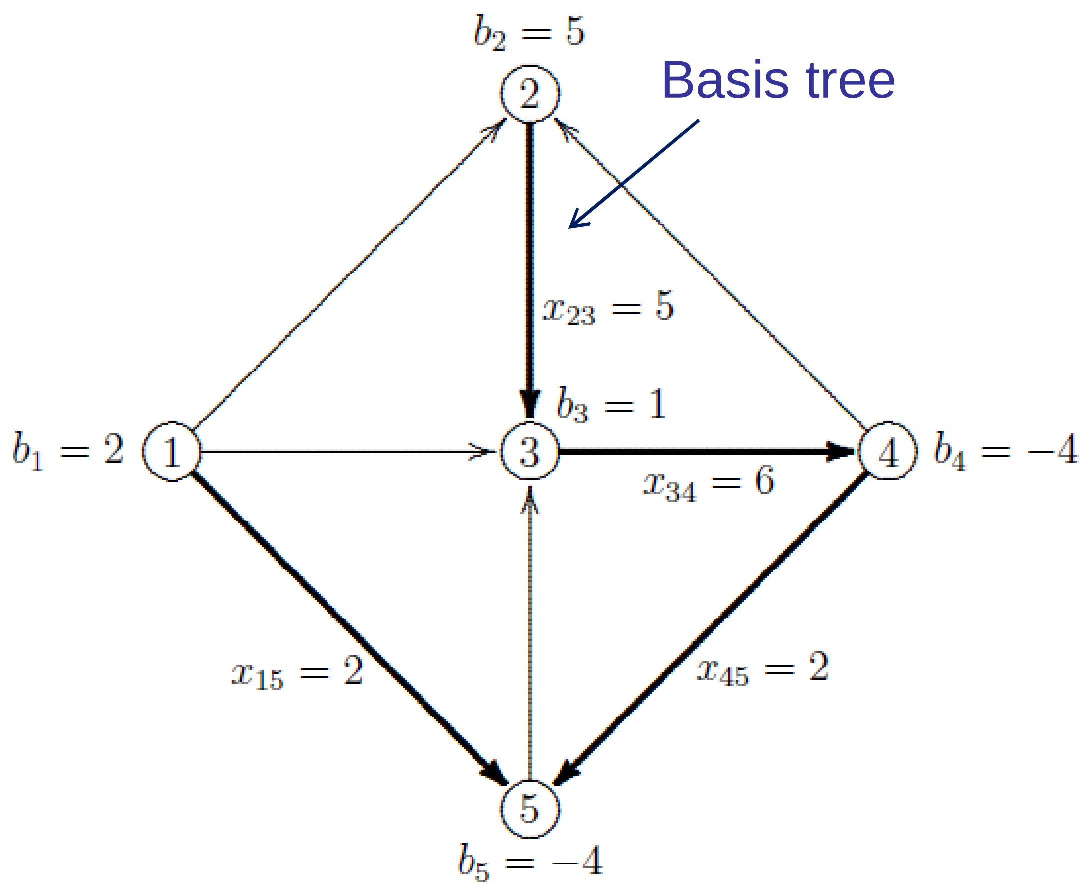
To evaluate r_{ij}, compute $u = c_B B^{-1}$ by solving triangular system $uB = c_B$.

Optimality conditions

To evaluate r_{ij} compute

$$u = c_B B^{-1}$$

by solving the triangular system $uB = c_B$



Equations to solve
(after fixing one u_i to, say, zero):

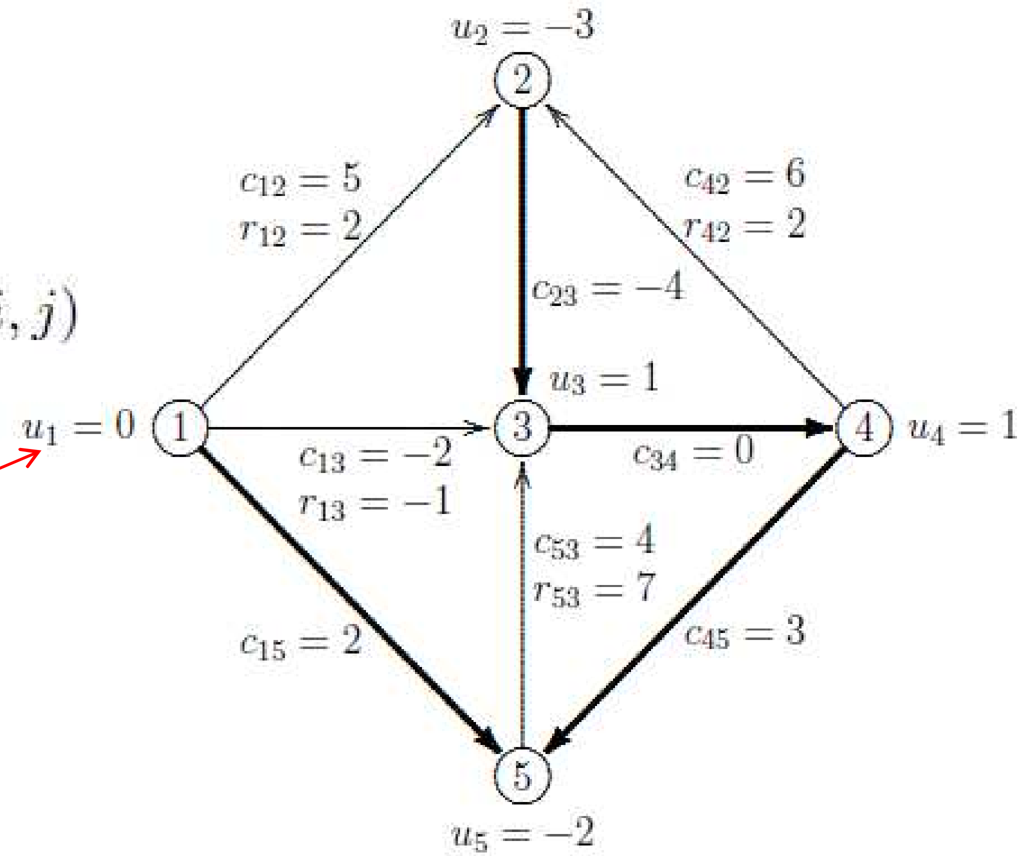
$$[u_1 \;\; u_2 \;\; u_3 \;\; u_4 \;\; u_5] \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ -1 & & & -1 \end{bmatrix} = [c_{15} \;\; c_{23} \;\; c_{34} \;\; c_{45}] = [2 \;\; -4 \;\; 0 \;\; 3]$$

Optimality conditions

Can solve

$$u_i - u_j = c_{ij}, \;\; \text{all basic arcs } (i, j)$$

directly on the network:



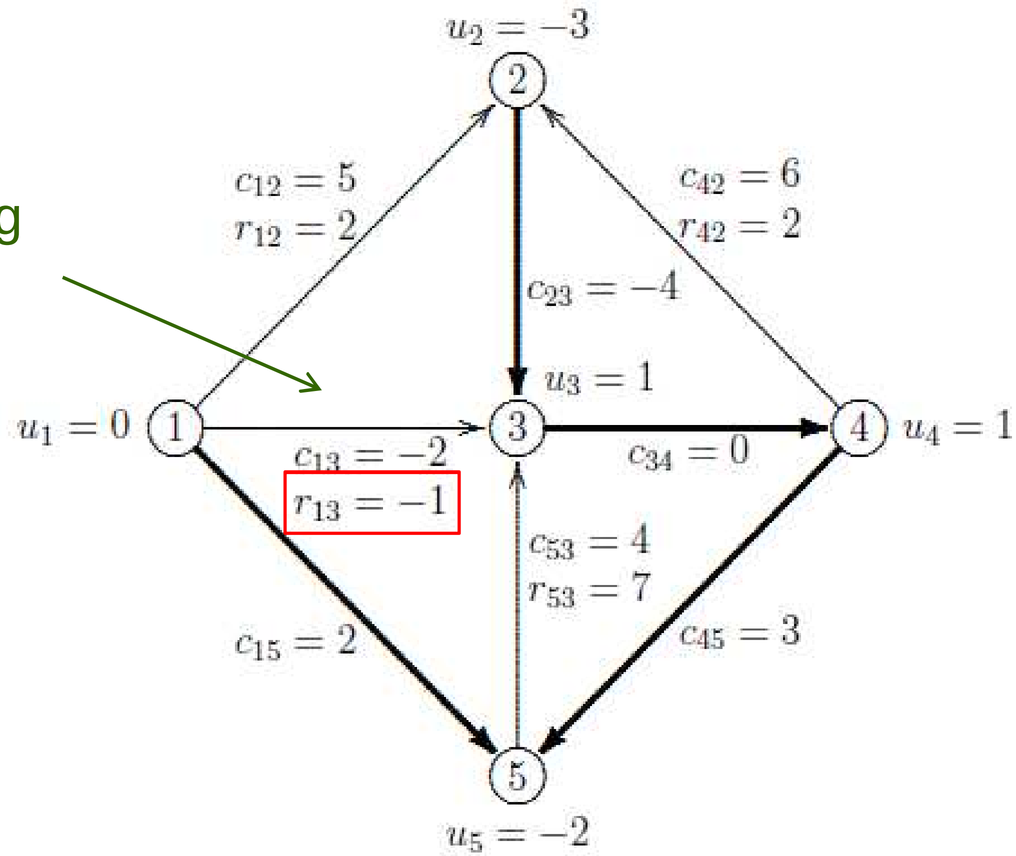
Fix this **potential** to zero, e.g.

Equations to solve
(after fixing one u_i to, say, zero):

$$[u_1 \;\; u_2 \;\; u_3 \;\; u_4 \;\; u_5] \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ -1 & & & -1 \end{bmatrix} = [c_{15} \;\; c_{23} \;\; c_{34} \;\; c_{45}] = [2 \;\; -4 \;\; 0 \;\; 3]$$

Optimality conditions

Can improve solution by adding arc with negative reduced cost to basis.



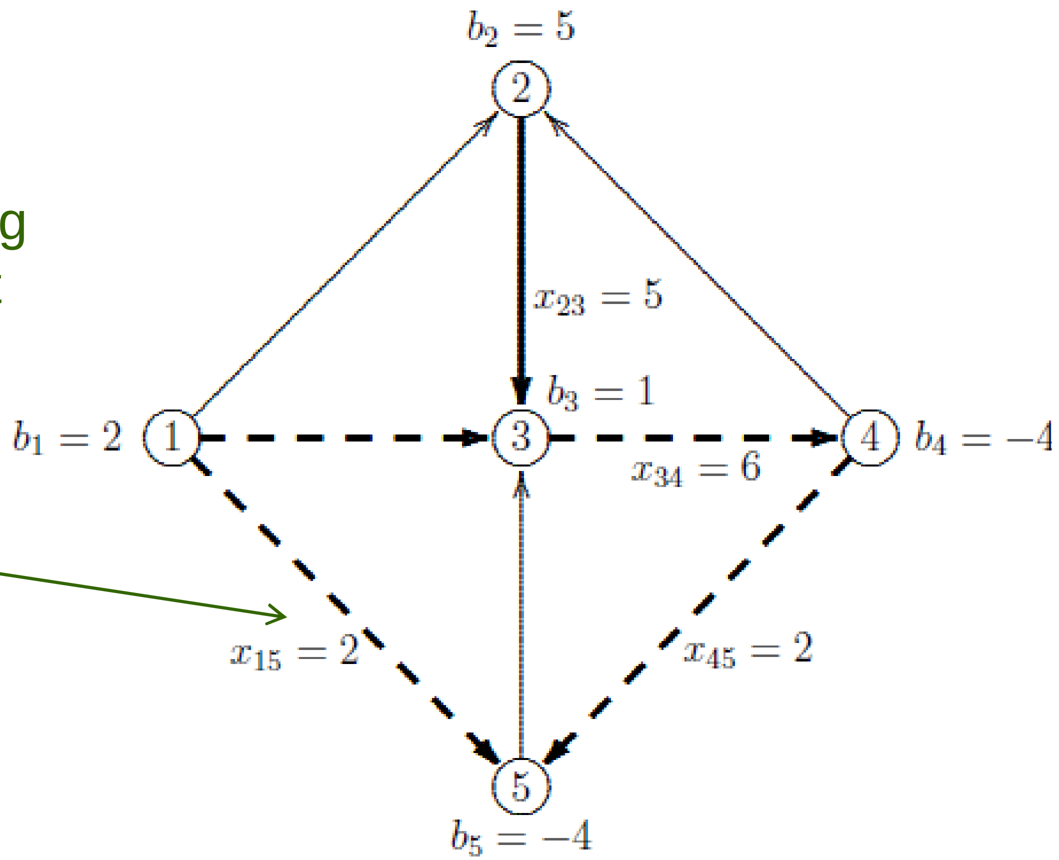
Reduced cost is

$r_{13} = c_{13} - u_1 + u_3 = -1$

Improvement step

Can improve solution by adding arc with negative reduced cost to basis.

This creates a cycle.

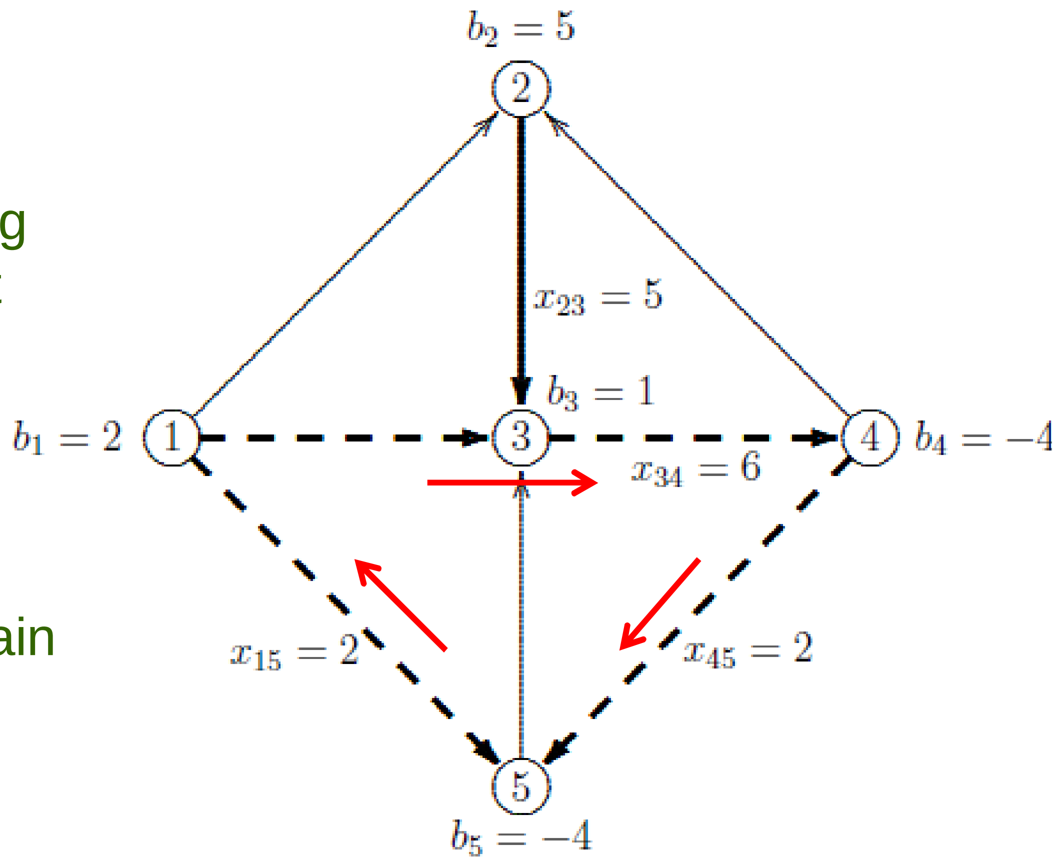


Improvement step

Can improve solution by adding arc with negative reduced cost to basis.

This creates a cycle.

Move flow around cycle to obtain next basic solution.



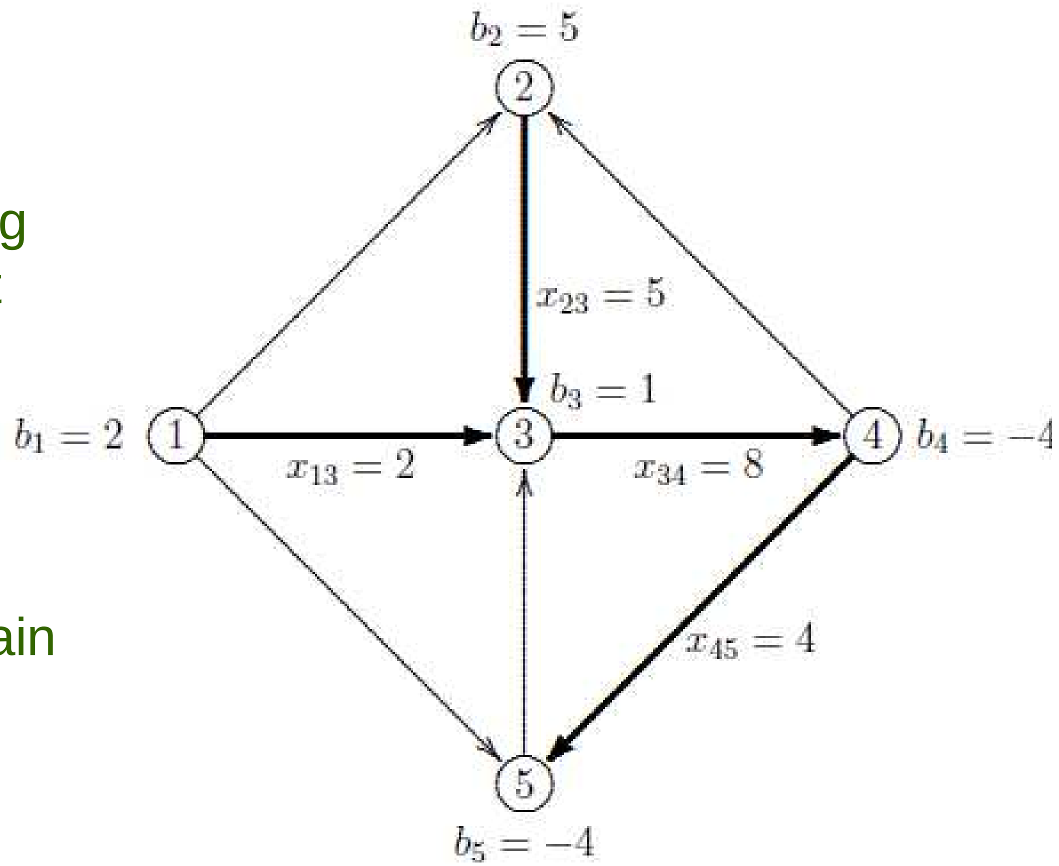
Improvement step

Can improve solution by adding arc with negative reduced cost to basis.

This creates a cycle.

Move flow around cycle to obtain next basic solution.

This is one step of the **network simplex method**.

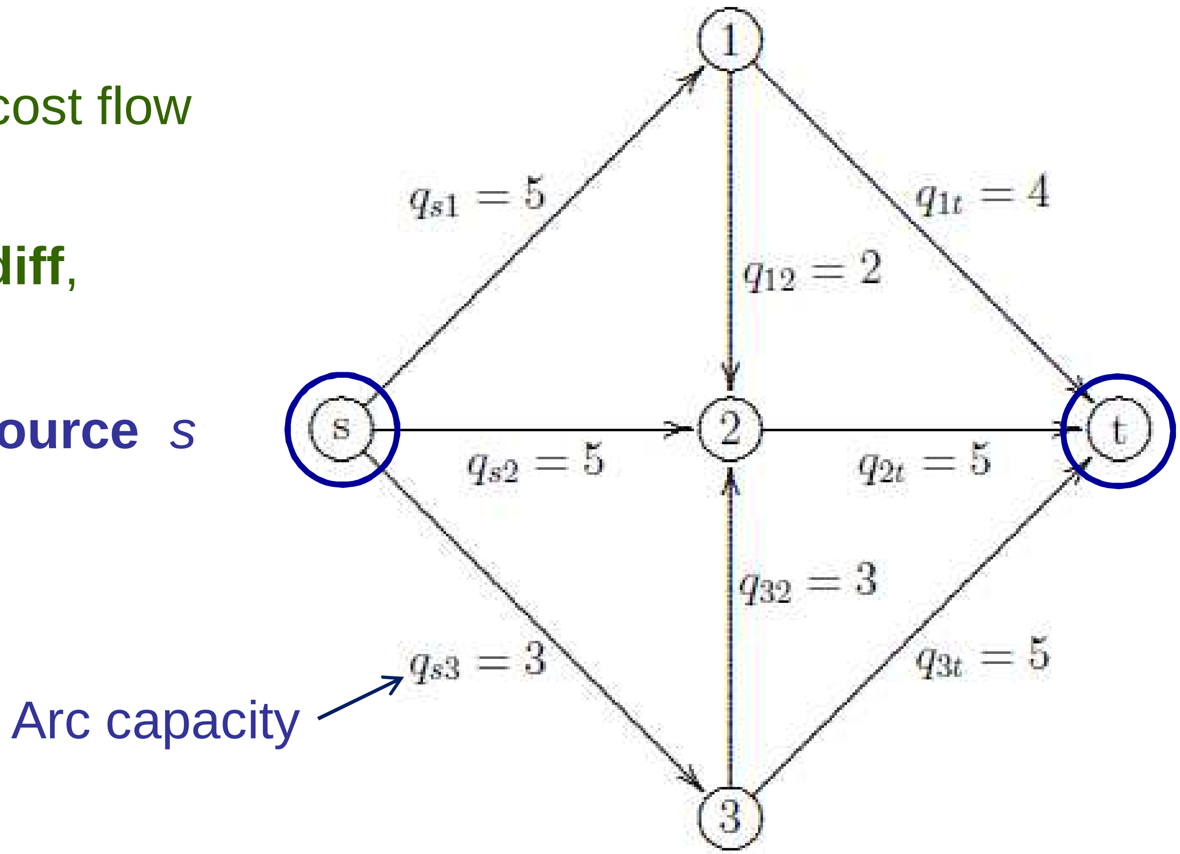


Max Flow Problem

Special case of max cost flow problem.

Useful for filtering **alldiff**, **cardinality**, etc.

Maximize flow from **source** *s* to **sink** *t* .



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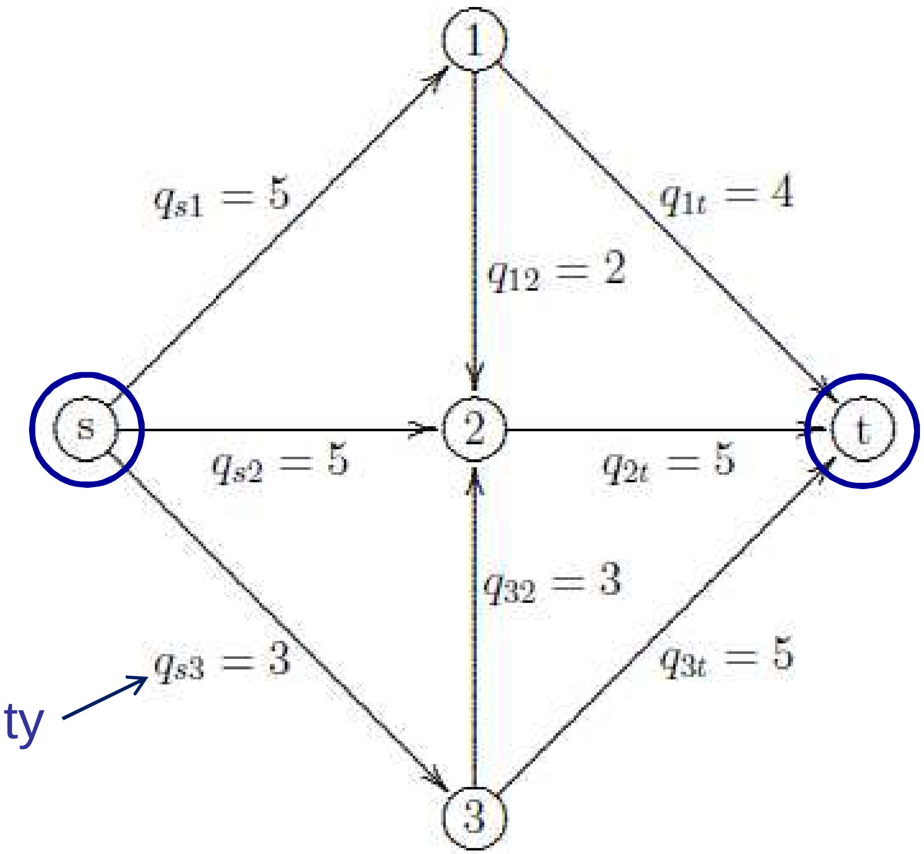


Arc capacity

In general,

$$\max \ x_{ts}$$

$$\sum_j x_{ij} - \sum_j x_{ji} = 0, \quad \text{all } i$$

$$0 \le x_{ij} \le q_{ij}, \quad \text{all } i, j$$

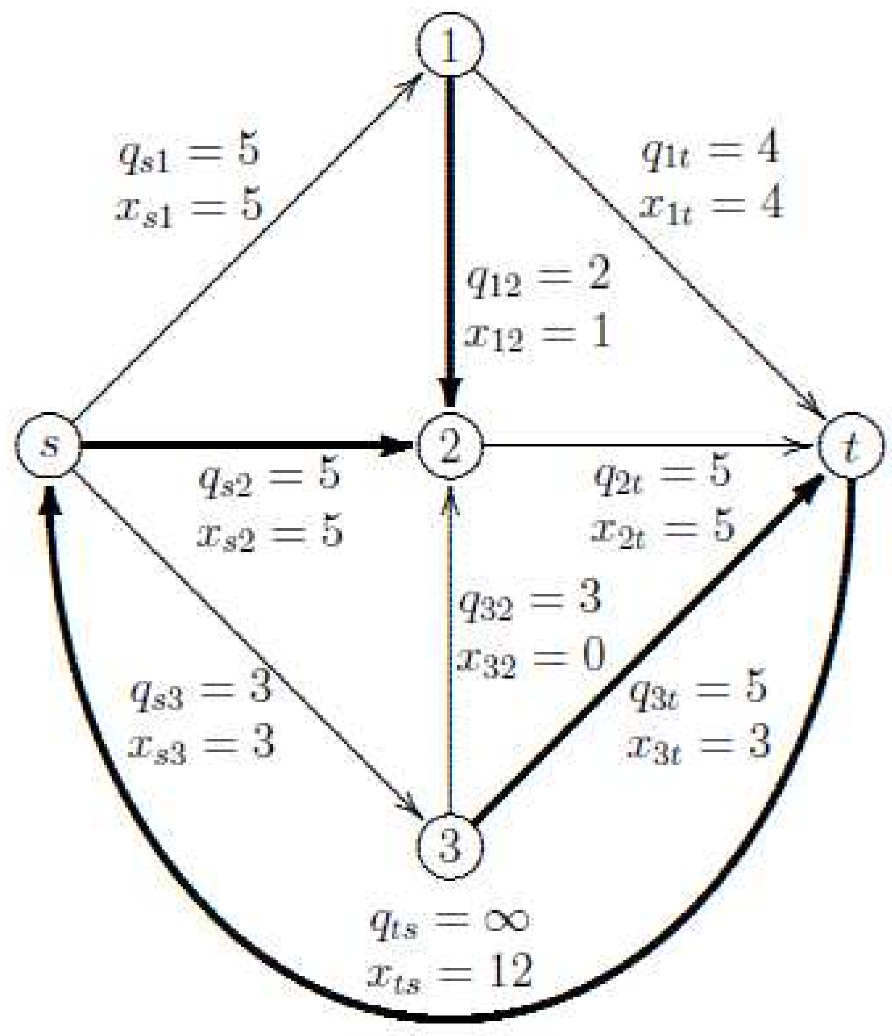
Max Flow Problem

Special case of max cost flow problem.

Formulation as max cost flow problem:

Cost is 1 on return arc, zero on other arcs.

Basic solution is shown.



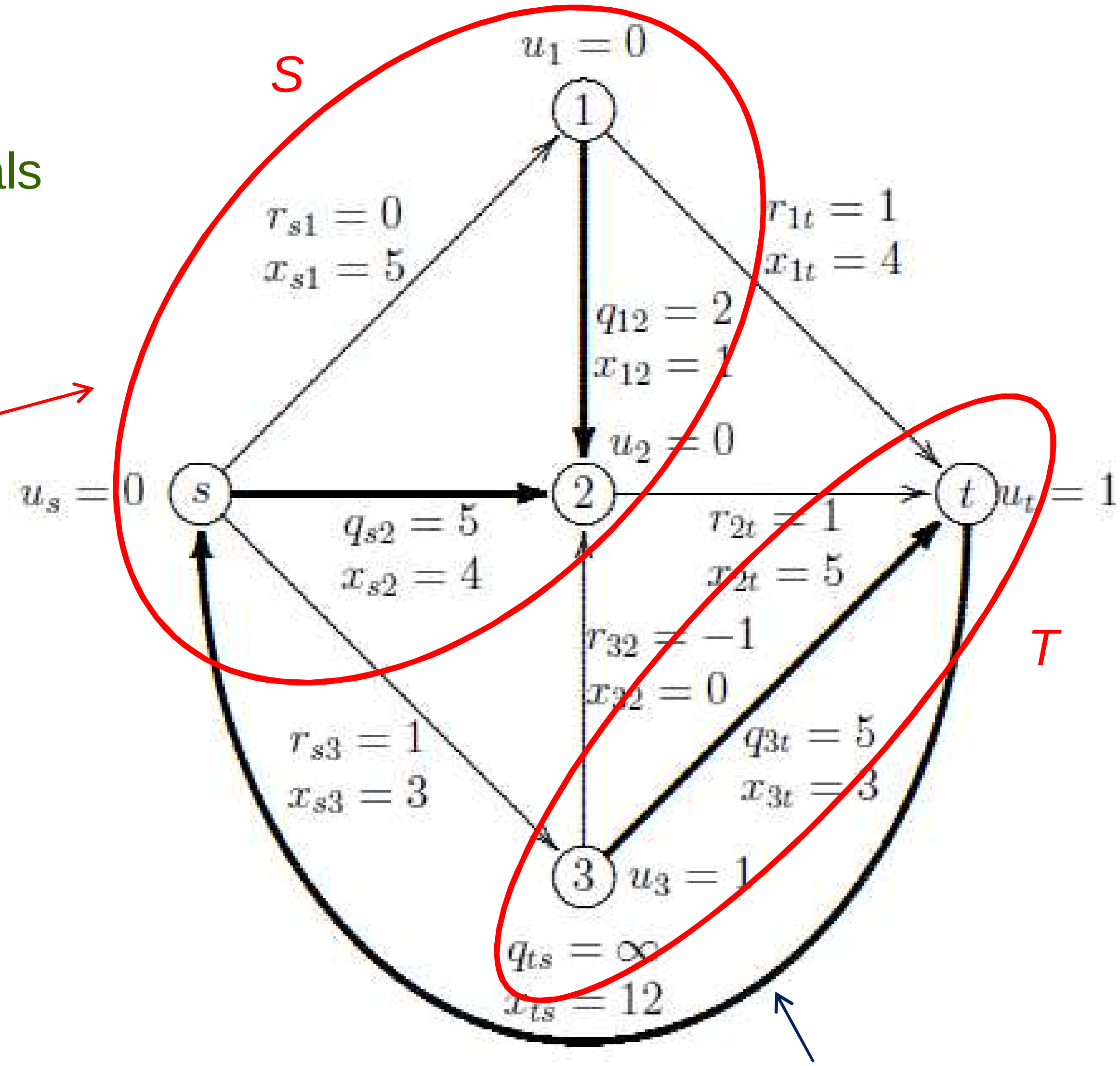
Max Flow Problem

Easy to compute potentials (dual variables).

This is an ***S-T*** **cut**

Potentials in *S* = 0

Potentials in *T* = 1



Cost = 1

Max Flow Problem

Easy to compute potentials (dual variables).

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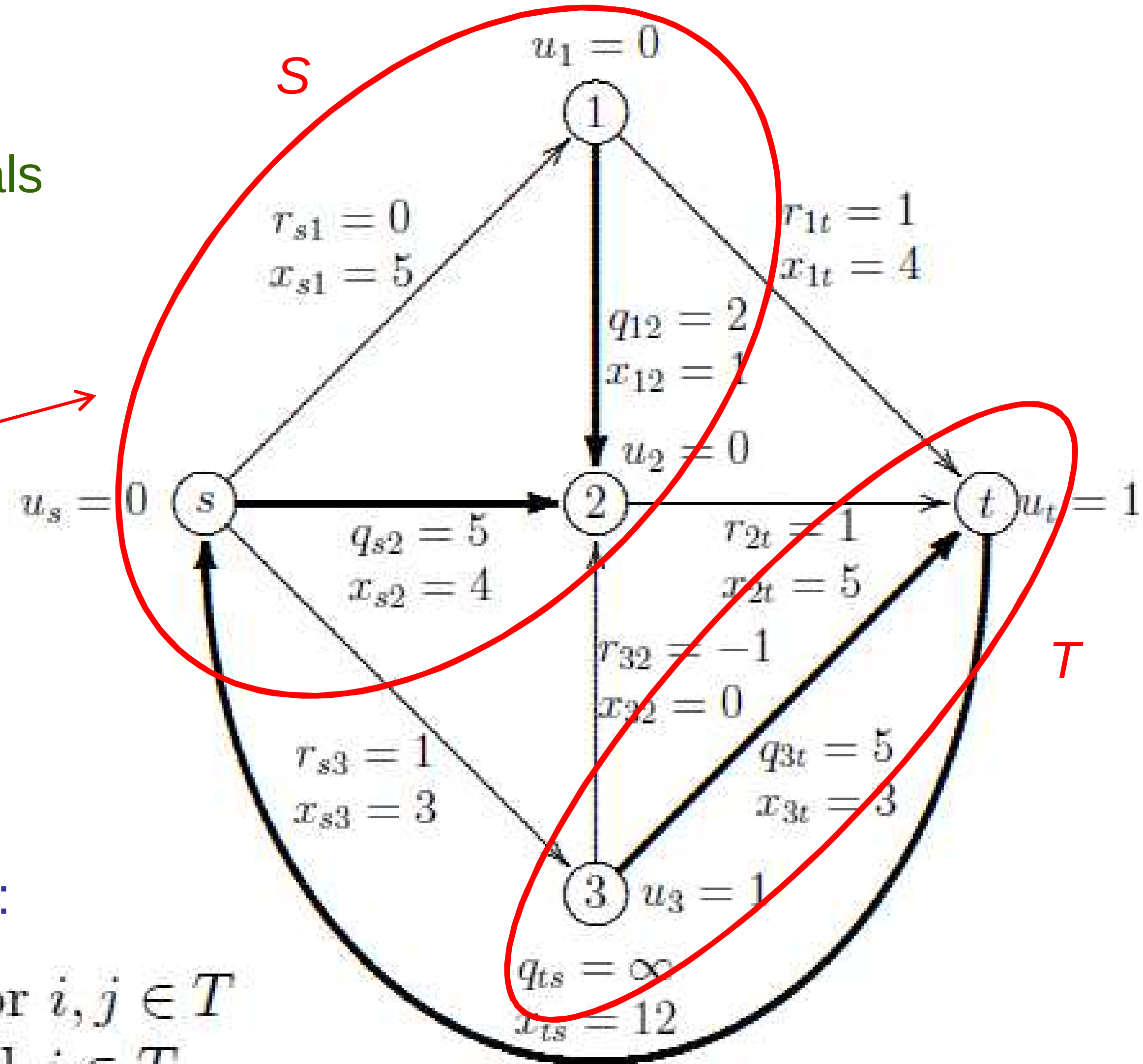


Potentials in $S = 0$

Potentials in $T = 1$

Reduced costs also easy:

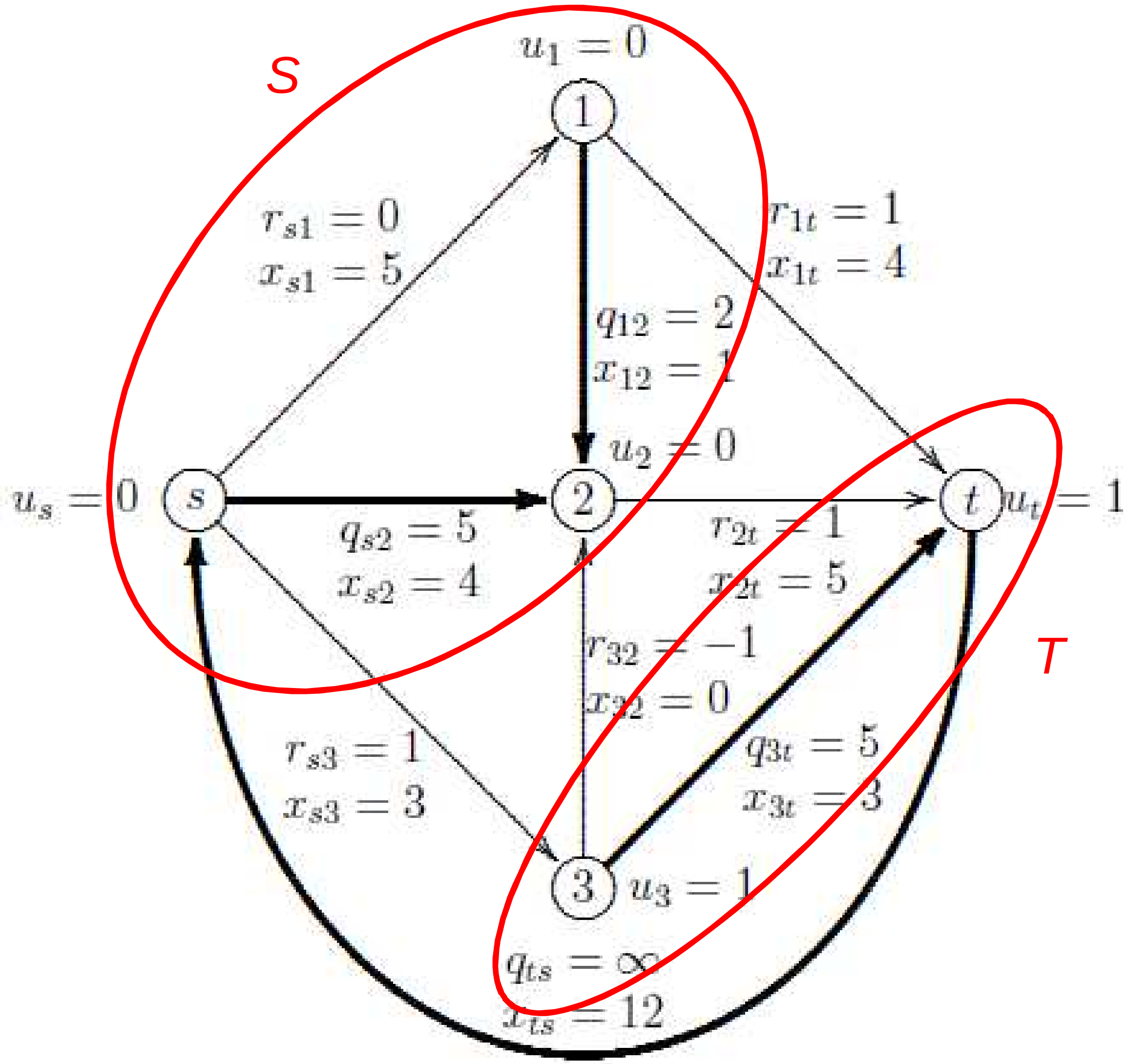
$$r_{ij} = \begin{cases} 0, \text{ if } i,j \in S \text{ or } i,j \in T \\ 1, \text{ if } i \in S \text{ and } j \in T \\ -1, \text{ if } i \in T \text{ and } j \in S \end{cases}$$

Max Flow Problem

So, basic solution is **optimal** if

Flows *S* →*T* are at capacity

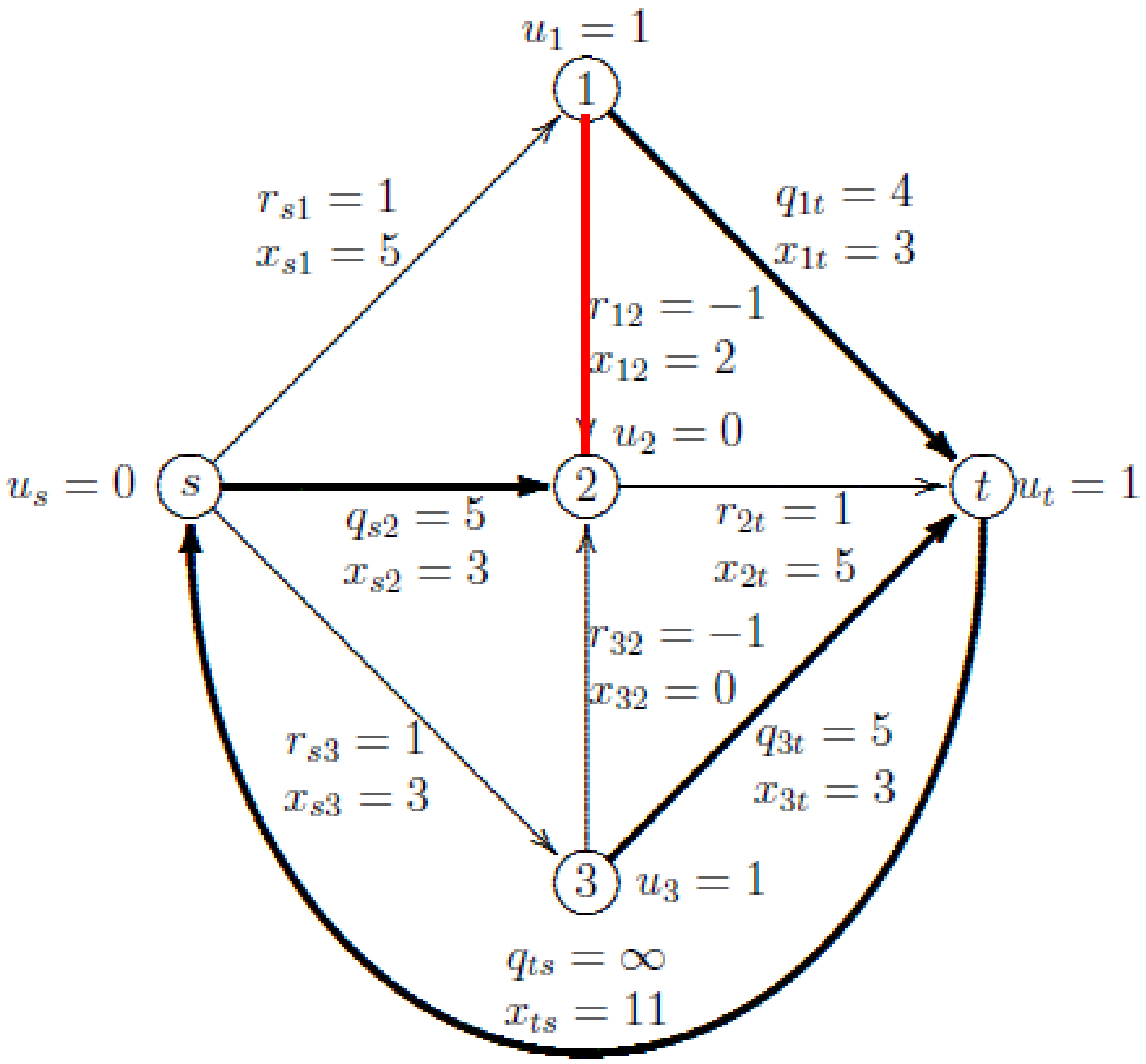
Flows *T* →*S* are zero



Improvement step

This basic solution is suboptimal.

Add nonzero *T-S* arc to the basis.



Improvement step

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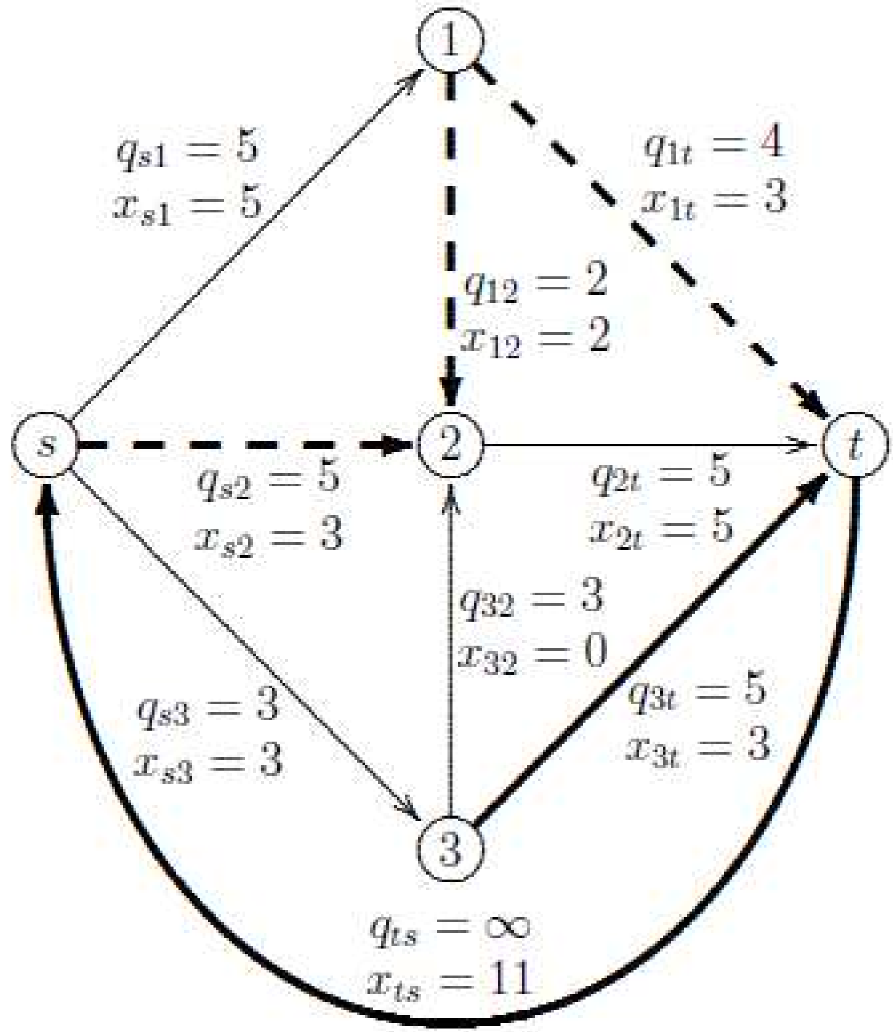
Add nonzero *T*-*S* arc to the basis.

This creates a cycle.



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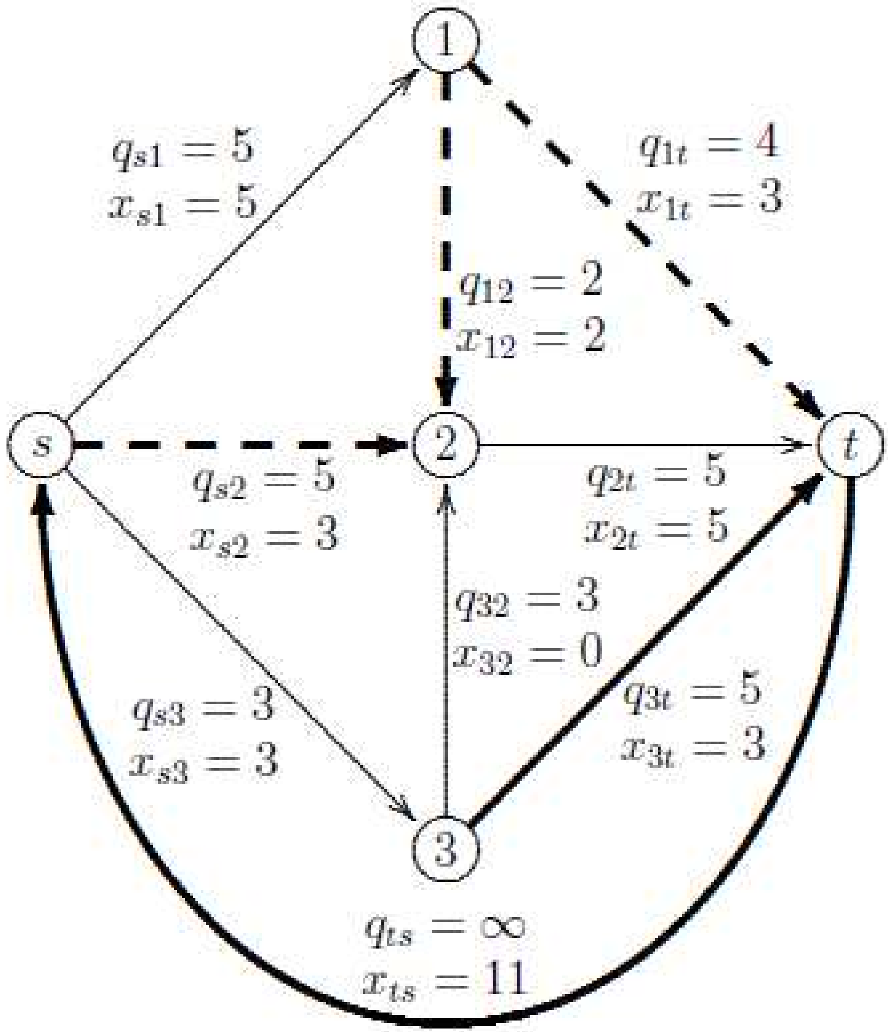
To increase total *s*-*t* flow:

Increase flow on forward arcs of dashed path, decrease on backward arcs.



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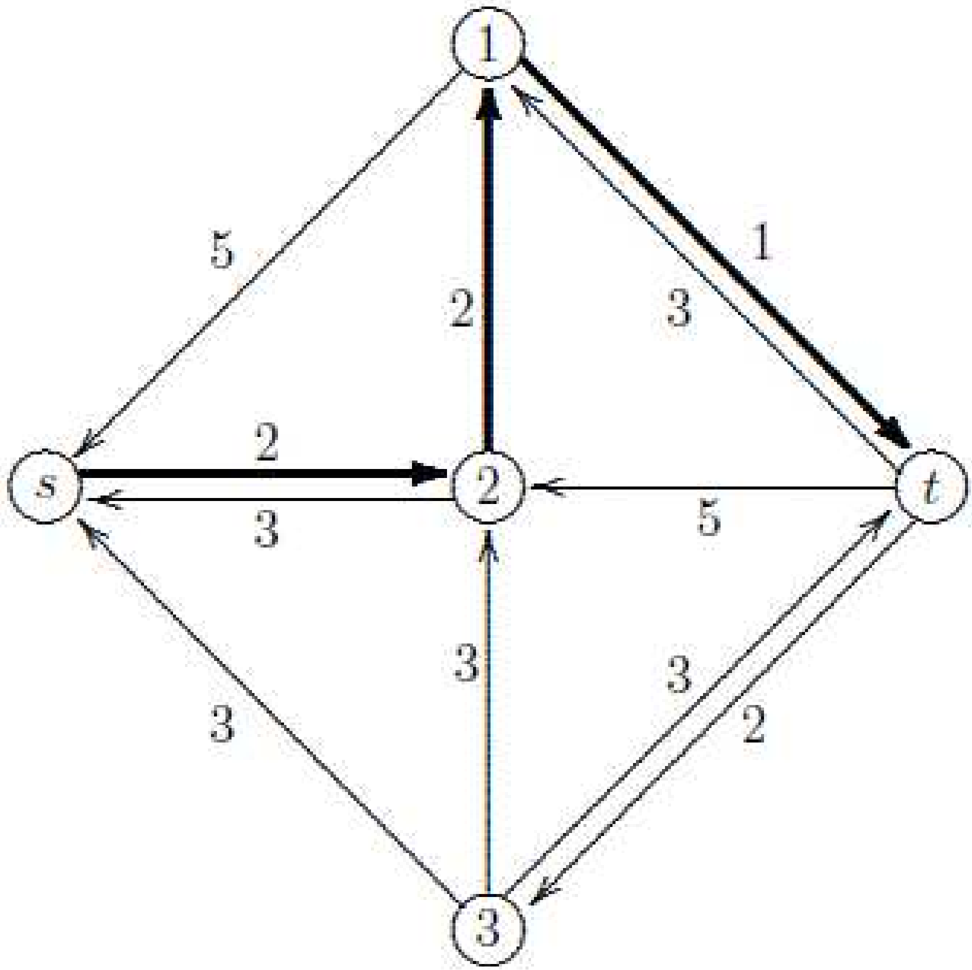
To increase total *s*-*t* flow:

Increase flow on forward arcs of dashed path, decrease on backward arcs.

Equivalently, increase flow on **augmenting path** of the **residual graph**.



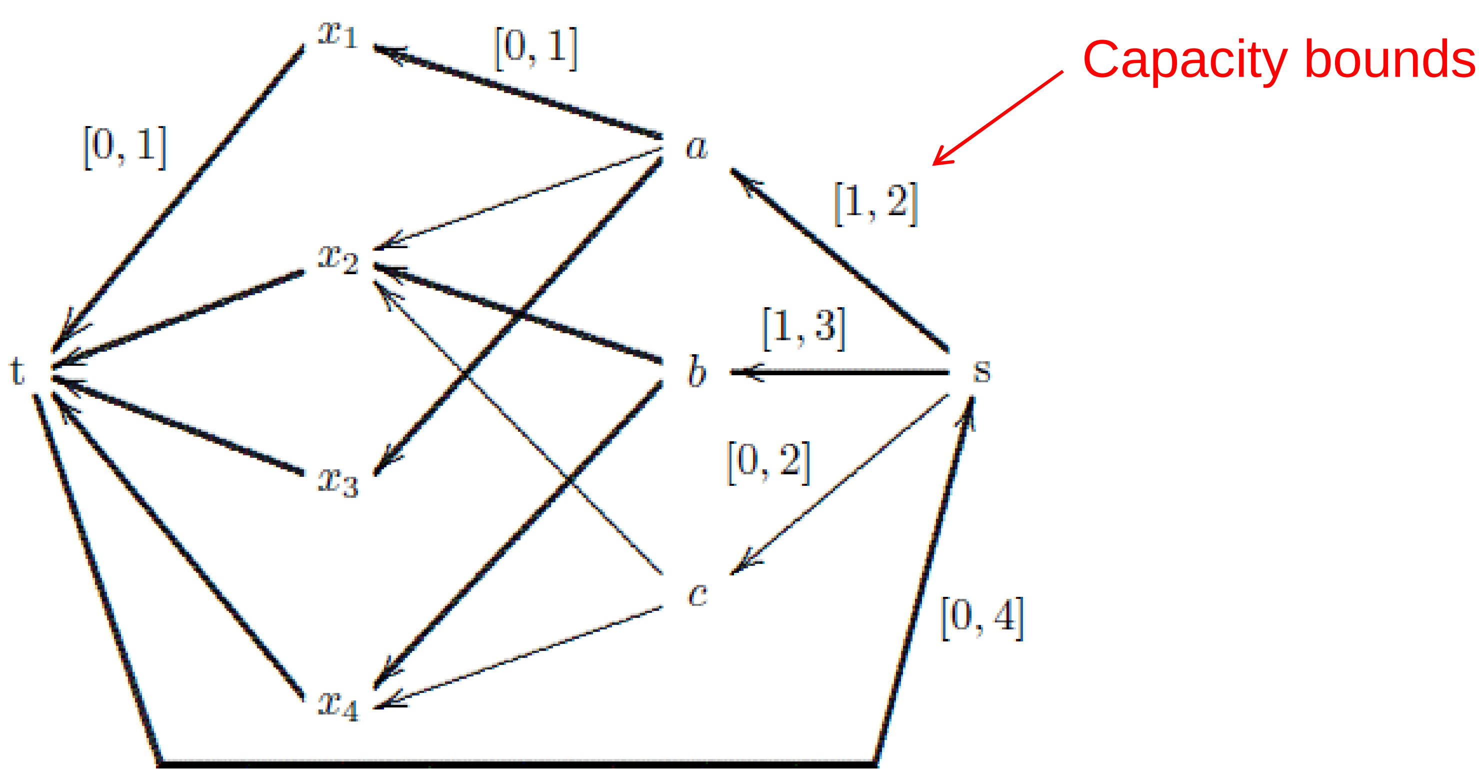
Filtering: Cardinality Constraint

Network flow model of

$$\text{cardinality}(\{x_1, x_2, x_3, x_4\} \mid (a, b, c), (1, 1, 0), (2, 3, 2))$$

with domains

$$D_{x_1} = D_{x_3} = \{a\},\ D_{x_2} = \{a, b, c\},\ D_{x_4} = \{b, c\}$$



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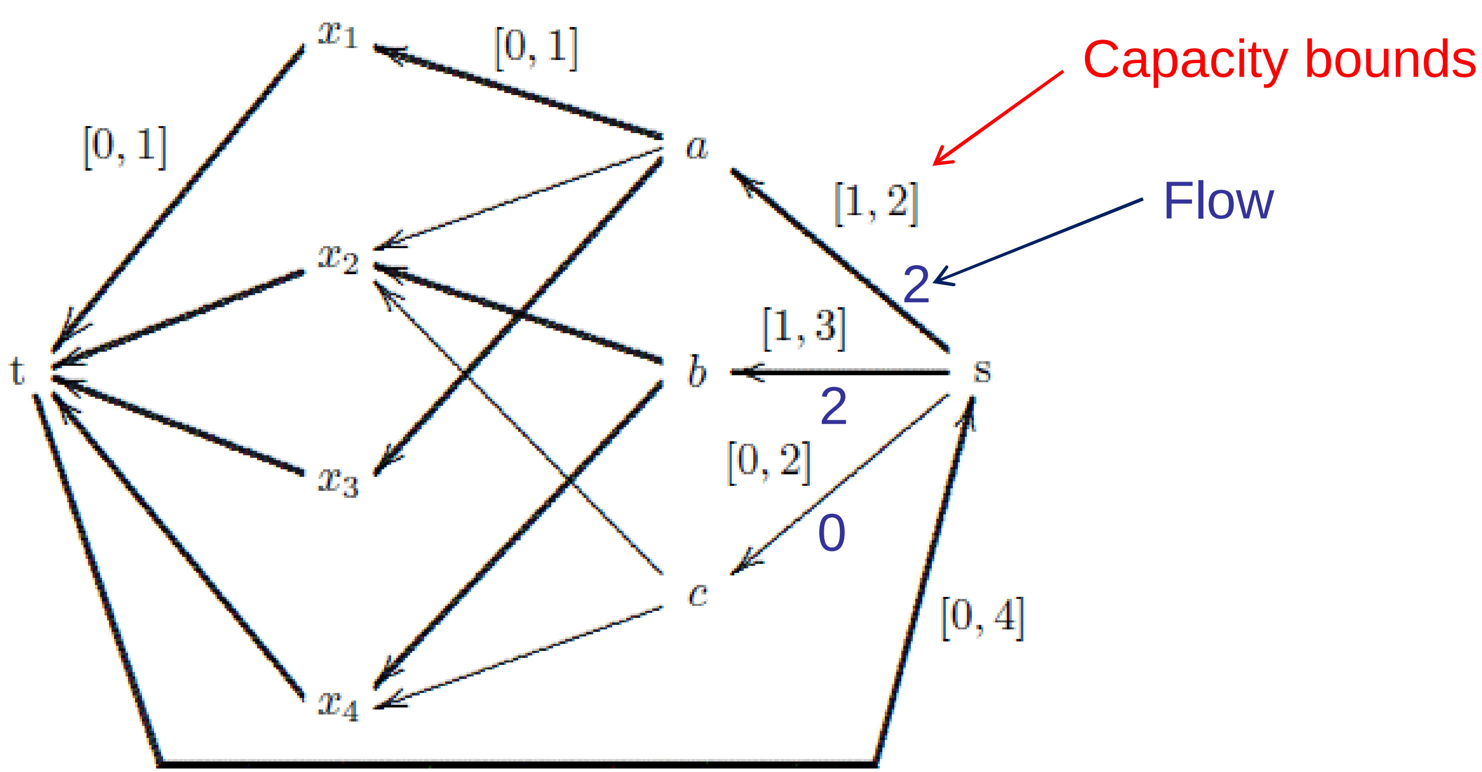

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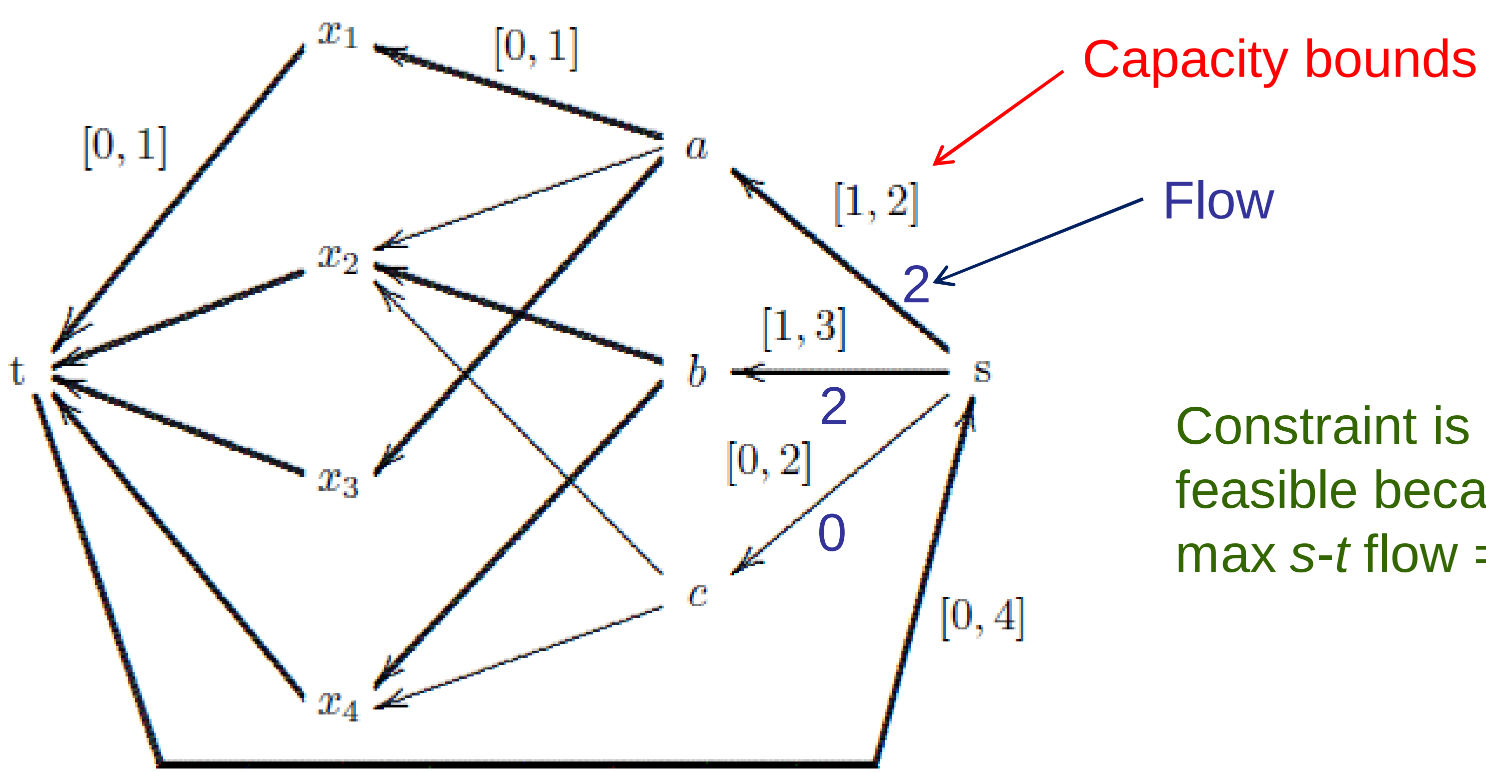
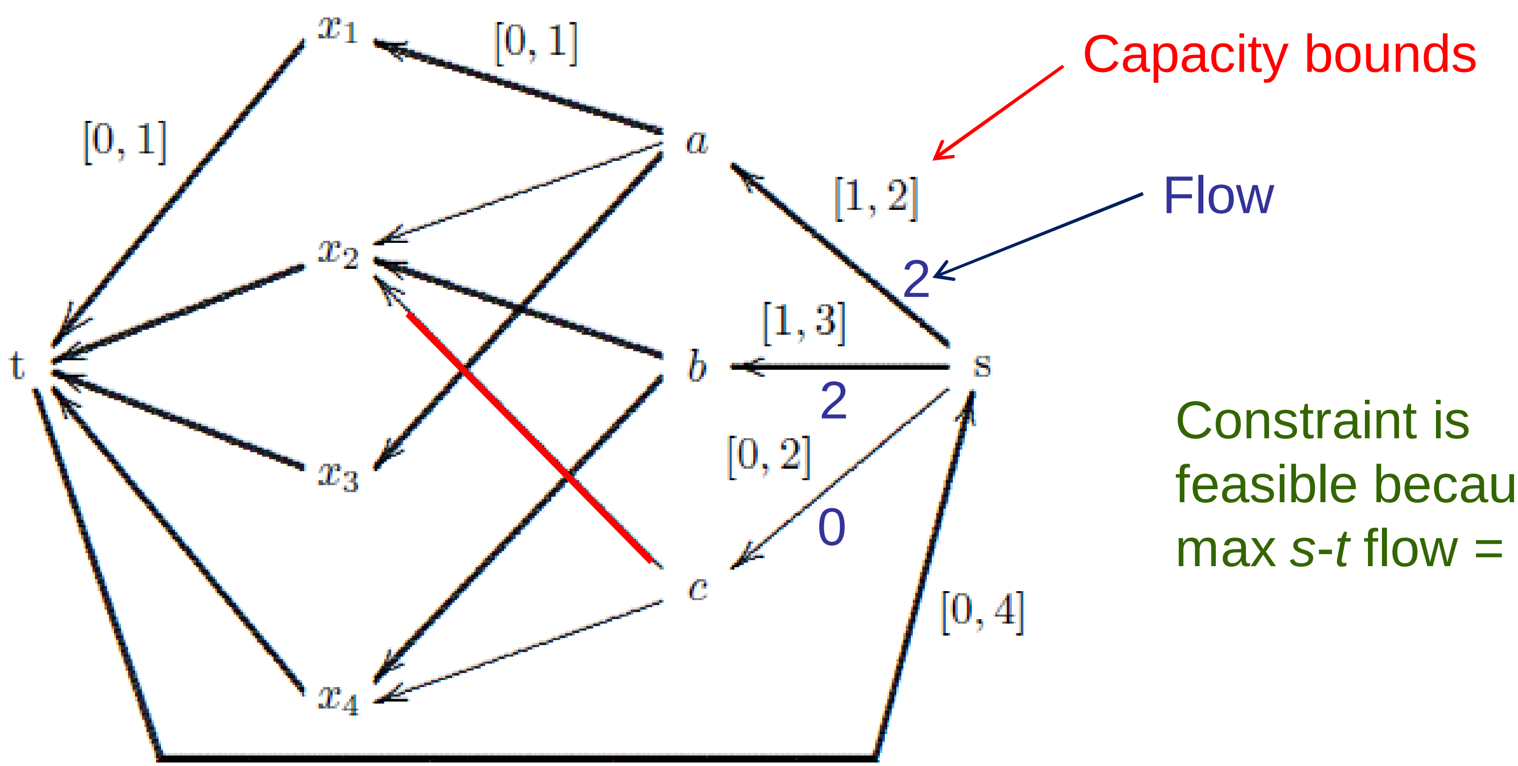


Constraint is feasible because max s-t flow = 4

Filtering: Cardinality Constraint

Network-based filtering achieves domain consistency.

Can remove *c* from domain of x_2 if max flow from *c* to x_2 is zero.



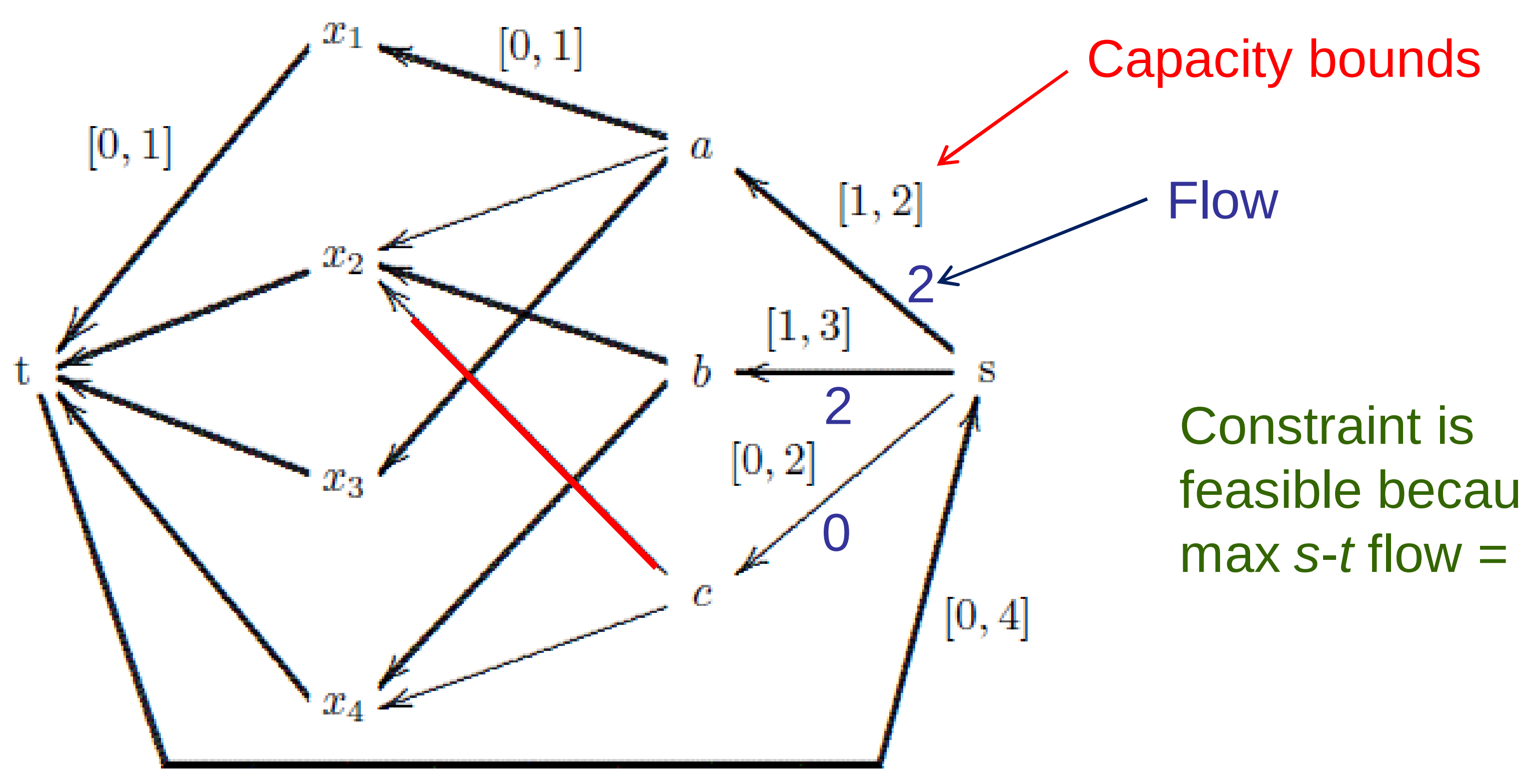
Constraint is feasible because max *s*-*t* flow = 4

Filtering: Cardinality Constraint

Network-based filtering achieves domain consistency.

Can remove c from domain of x_2 if max flow from c to x_2 is zero.

But zero is **not** max flow. There is an augmenting path from x_2 to c in the residual graph.



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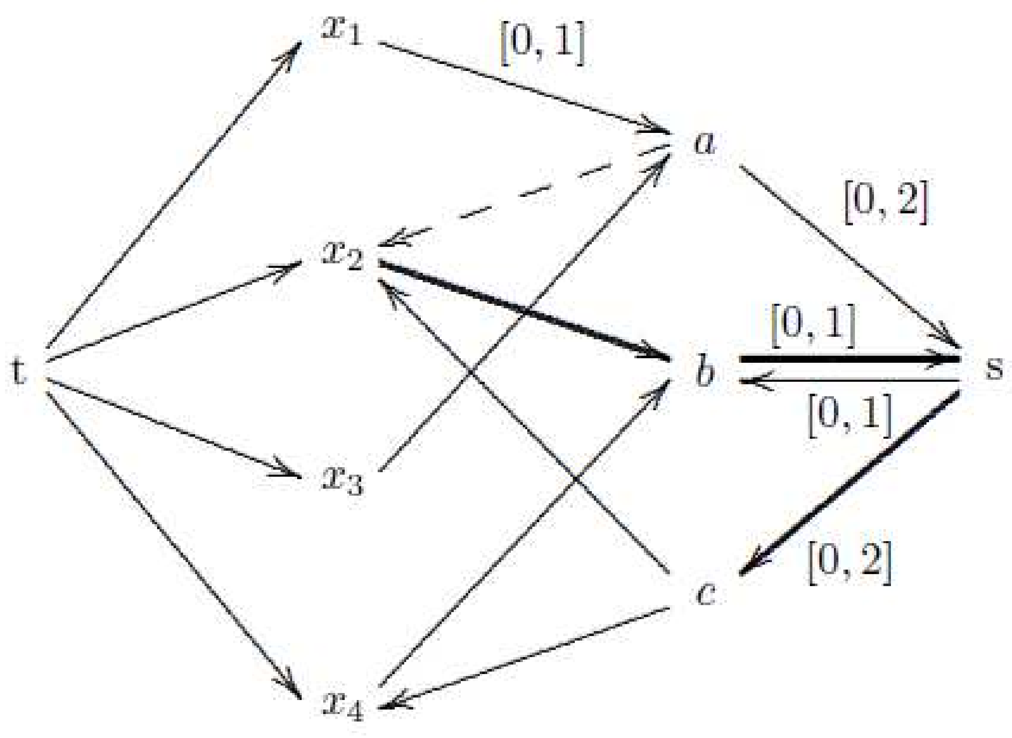

Residual graph

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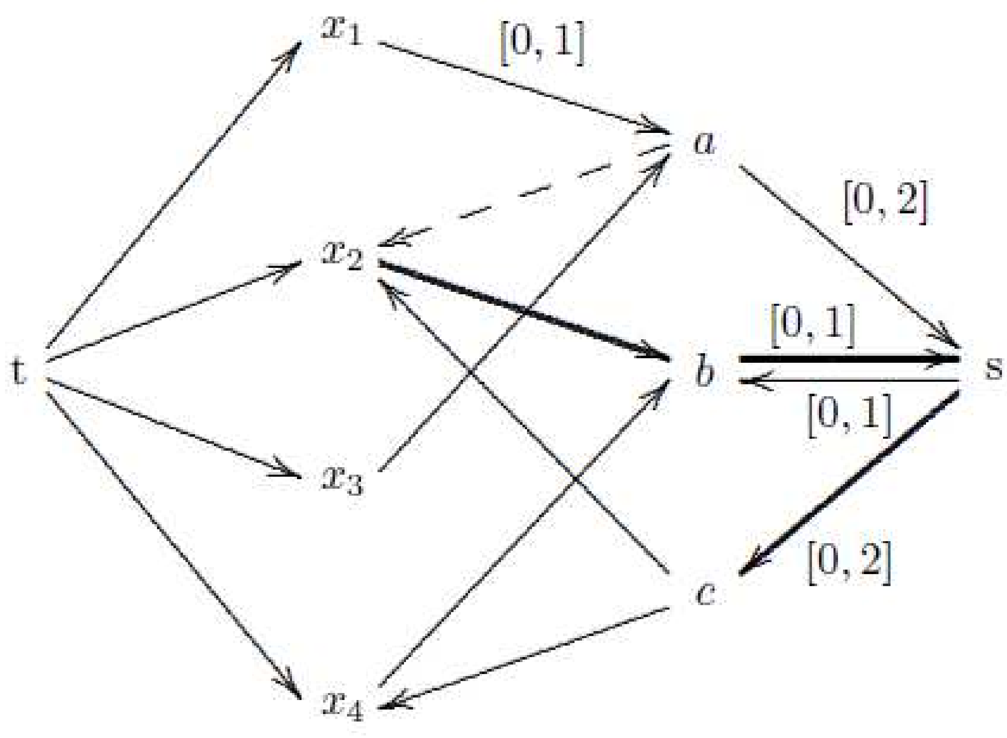
But zero is **not** max flow. There is an augmenting path from x_2 to *c* in the residual graph.

However, we can remove *a* from domain of x_2 (no augmenting path).

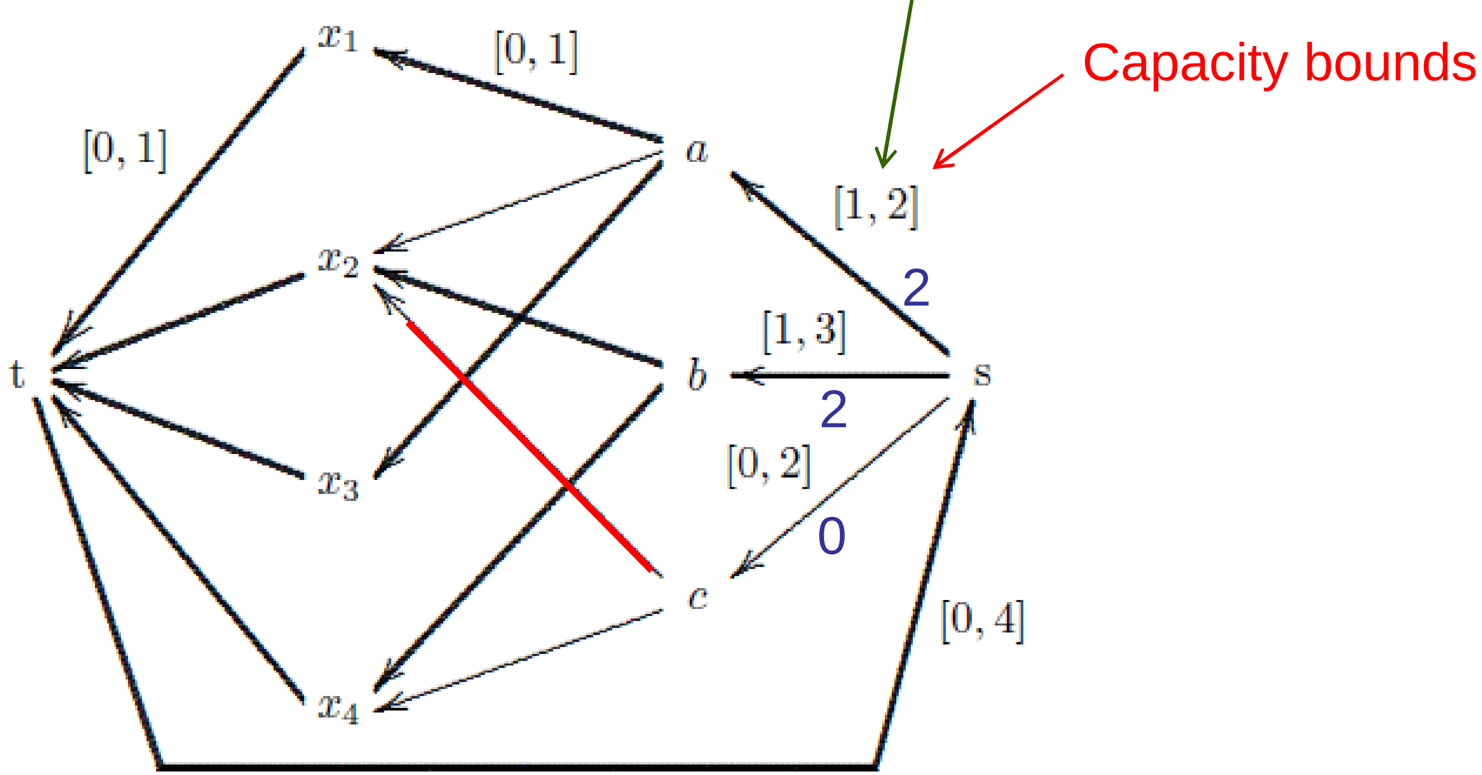


Residual graph

Filtering: Alldiff

Alldiff is a special case in which these capacities are [0,1].

(Max cardinality bipartite matching)



Filtering: Sequence

The sequence constraint has several polytime filters that achieve domain consistency:

- Cumulative sums (also filters genSequence)
- Network flow model
- Decomposition and propagation (based on Berge acyclicity of constraint hypergraph).

We will develop the **network flow model**.

Filtering: Sequence

Consider constraint $\text{sequence}\,((y_1,\ldots,y_7)\,|\,3,\{1\},\ell,u)$

That is, every stretch of 3 variables y_i must contain at least ℓ and at most u 1's.

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IP formulation is $\ell \le y_{j-2} + y_{j-1} + y_j \le u, \quad j = 3, \ldots, 7$

or

$$
\begin{bmatrix}
1&1&1&&&&&-1&&&&&&&&& \\
1&1&1&&&&&&1&&&&&&&& \\
&1&1&1&&&&&&-1&&&&&&& \\
&1&1&1&&&&&&&1&&&&&& \\
&&1&1&1&&&&&&&-1&&&&& \\
&&1&1&1&&&&&&&&1&&&& \\
&&&1&1&1&&&&&&&&-1&&& \\
&&&1&1&1&&&&&&&&&1&& \\
&&&&1&1&1&&&&&&&&&-1& \\
&&&&1&1&1&&&&&&&&&&1
\end{bmatrix}
\begin{bmatrix} y_1 \\ \vdots \\ y_7 \\ w_3 \\ z_3 \\ \vdots \\ w_7 \\ z_7 \end{bmatrix}
=
\begin{bmatrix} \ell \\ u \\ \ell \\ u \\ \ell \\ u \\ \ell \\ u \\ \ell \\ u \end{bmatrix}
$$

Filtering: Sequence

The transpose of the matrix has the **consecutive ones property**.

We will see later that it is therefore **totally unimodular** and can be solved as an LP (all LP solutions are integral).

$$
\begin{bmatrix}
1 & 1 & 1 & & & & & -1 & & & & & & & & & \\
1 & 1 & 1 & & & & & & 1 & & & & & & & & \\
 & 1 & 1 & 1 & & & & & & -1 & & & & & & & \\
 & 1 & 1 & 1 & & & & & & & 1 & & & & & & \\
 & & 1 & 1 & 1 & & & & & & & -1 & & & & & \\
 & & 1 & 1 & 1 & & & & & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & & & & & & -1 & & & \\
 & & & 1 & 1 & 1 & & & & & & & & & 1 & & \\
 & & & & 1 & 1 & 1 & & & & & & & & & -1 & \\
 & & & & 1 & 1 & 1 & & & & & & & & & & 1
\end{bmatrix}
\begin{bmatrix} y_1 \\ \vdots \\ y_7 \\ w_3 \\ z_3 \\ \vdots \\ w_7 \\ z_7 \end{bmatrix}
=
\begin{bmatrix} \ell \\ u \\ \ell \\ u \\ \ell \\ u \\ \ell \\ u \\ \ell \\ u \end{bmatrix}
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In fact, it can be solved and filtered as a **network flow problem**.

$$
\left[\begin{array}{ccccccccccccccccc}
1 & 1 & 1 & & & & & -1 & & & & & & & & & \\
1 & 1 & 1 & & & & & & 1 & & & & & & & & \\
 & 1 & 1 & 1 & & & & & & -1 & & & & & & & \\
 & 1 & 1 & 1 & & & & & & & 1 & & & & & & \\
 & & 1 & 1 & 1 & & & & & & & -1 & & & & & \\
 & & 1 & 1 & 1 & & & & & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & & & & & & -1 & & & \\
 & & & 1 & 1 & 1 & & & & & & & & & 1 & & \\
 & & & & 1 & 1 & 1 & & & & & & & & & -1 & \\
 & & & & 1 & 1 & 1 & & & & & & & & & & 1
\end{array}\right]
\left[\begin{array}{c} y_1 \\ \vdots \\ y_7 \\ w_3 \\ z_3 \\ \vdots \\ w_7 \\ z_7 \end{array}\right]
=
\left[\begin{array}{c} \ell \\ u \\ \ell \\ u \\ \ell \\ u \\ \ell \\ u \\ \ell \\ u \end{array}\right]
$$

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Subtract each row from the next (after adding row of zeros):

$$
\begin{bmatrix}
1 & 1 & 1 & & & & & -1 & & & & & & & & & \\
1 & 1 & 1 & & & & & & 1 & & & & & & & & \\
& 1 & 1 & 1 & & & & & & -1 & & & & & & & \\
& 1 & 1 & 1 & & & & & & & 1 & & & & & & \\
& & 1 & 1 & 1 & & & & & & & -1 & & & & & \\
& & 1 & 1 & 1 & & & & & & & & 1 & & & & \\
& & & 1 & 1 & 1 & & & & & & & & -1 & & & \\
& & & 1 & 1 & 1 & & & & & & & & & 1 & & \\
& & & & 1 & 1 & 1 & & & & & & & & & -1 & \\
& & & & 1 & 1 & 1 & & & & & & & & & & 1
\end{bmatrix}
\begin{bmatrix} y_1 \\ \vdots \\ y_7 \\ w_3 \\ z_3 \\ \vdots \\ w_7 \\ z_7 \end{bmatrix}
=
\begin{bmatrix} \ell \\ u \\ \ell \\ u \\ \ell \\ u \\ \ell \\ u \\ \ell \\ u \end{bmatrix}
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 & & & & & & & 1 & 1 & & & & & & & & \\
-1 & & & 1 & & & & & -1 & -1 & & & & & & & \\
 & & & & & & & & & 1 & 1 & & & & & & \\
 & -1 & & & 1 & & & & & & -1 & -1 & & & & & \\
 & & & & & & & & & & & 1 & 1 & & & & \\
 & & -1 & & & 1 & & & & & & & -1 & -1 & & & \\
 & & & & & & & & & & & & & 1 & 1 & & \\
 & & & -1 & & & 1 & & & & & & & & -1 & -1 & \\
 & & & & & & & & & & & & & & & 1 & 1 \\
 & & & & -1 & -1 & -1 & & & & & & & & & & -1
\end{bmatrix}
\begin{bmatrix} y_1 \\ \vdots \\ y_7 \\ w_3 \\ z_3 \\ \vdots \\ w_7 \\ z_7 \end{bmatrix}
=
\begin{bmatrix} \ell \\ u-\ell \\ \ell-u \\ u-\ell \\ \ell-u \\ u-\ell \\ \ell-u \\ u-\ell \\ \ell-u \\ u-\ell \\ -u \end{bmatrix}
\begin{array}{c} (b_3) \\ (a_3) \\ (b_4) \\ (a_4) \\ (b_5) \\ (a_5) \\ (b_6) \\ (a_6) \\ (b_7) \\ (a_7) \\ (b_8) \end{array}
$$

Filtering: Sequence

This is a network flow problem.
The network is…

$$
\begin{bmatrix}
1 & 1 & 1 & & & & & -1 & & & & & & & & & \\
 & & & & & & & 1 & 1 & & & & & & & & \\
-1 & & & 1 & & & & & -1 & -1 & & & & & & & \\
 & & & & & & & & & 1 & 1 & & & & & & \\
 & -1 & & & 1 & & & & & & -1 & -1 & & & & & \\
 & & & & & & & & & & & 1 & 1 & & & & \\
 & & -1 & & & 1 & & & & & & & -1 & -1 & & & \\
 & & & & & & & & & & & & & 1 & 1 & & \\
 & & & -1 & & & 1 & & & & & & & & -1 & -1 & \\
 & & & & & & & & & & & & & & & 1 & 1 \\
 & & & & -1 & -1 & -1 & & & & & & & & & & -1
\end{bmatrix}
\begin{bmatrix} y_1 \\ \vdots \\ y_7 \\ w_3 \\ z_3 \\ \vdots \\ w_7 \\ z_7 \end{bmatrix}
=
\begin{array}{l}
\begin{bmatrix} \ell \\ u-\ell \\ \ell-u \\ u-\ell \\ \ell-u \\ u-\ell \\ \ell-u \\ u-\ell \\ \ell-u \\ u-\ell \\ -u \end{bmatrix}
\end{array}
\begin{array}{c} (b_3) \\ (a_3) \\ (b_4) \\ (a_4) \\ (b_5) \\ (a_5) \\ (b_6) \\ (a_6) \\ (b_7) \\ (a_7) \\ (b_8) \end{array}
$$

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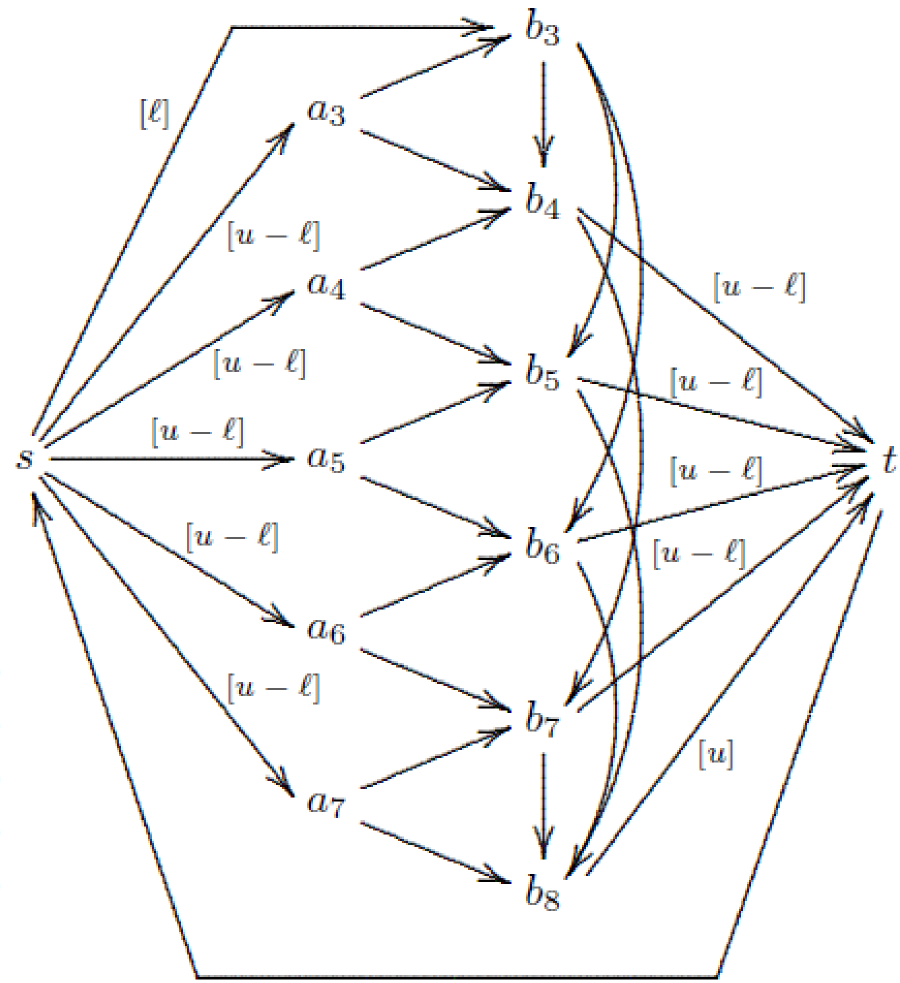
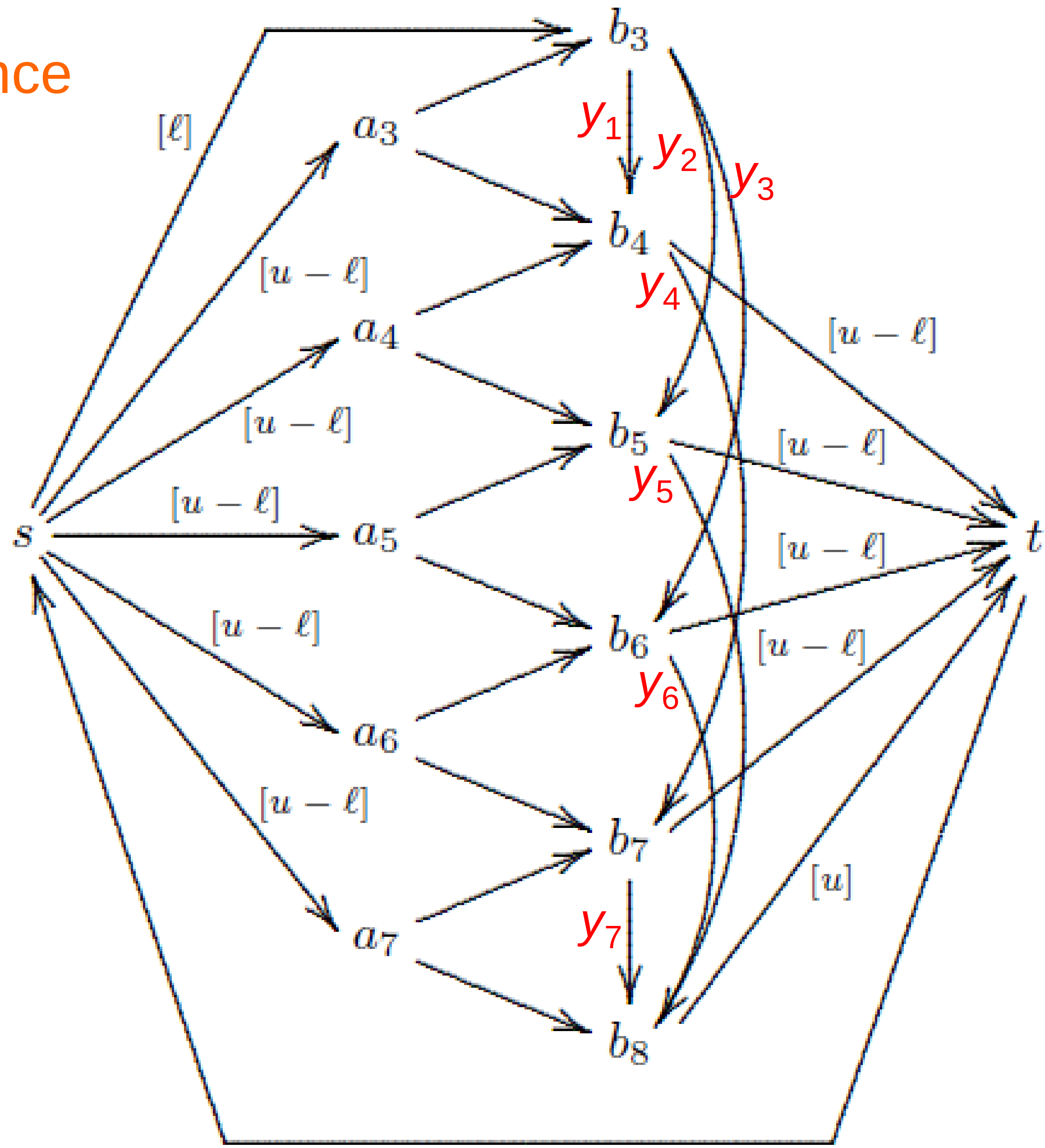
$$
\begin{bmatrix}
1 & 1 & 1 & & & & & -1 & & & & & & & & & \\
 & & & & & & & 1 & 1 & & & & & & & & \\
-1 & & & 1 & & & & & -1 & -1 & & & & & & & \\
 & & & & & & & & & 1 & 1 & & & & & & \\
 & -1 & & & 1 & & & & & & -1 & -1 & & & & & \\
 & & & & & & & & & & & 1 & 1 & & & & \\
 & & -1 & & & 1 & & & & & & & -1 & -1 & & & \\
 & & & & & & & & & & & & & 1 & 1 & & \\
 & & & -1 & & & 1 & & & & & & & & -1 & -1 & \\
 & & & & & & & & & & & & & & & 1 & 1 \\
 & & & & -1 & -1 & -1 & & & & & & & & & & -1
\end{bmatrix}
\begin{bmatrix} y_1 \\ \vdots \\ y_7 \\ w_3 \\ z_3 \\ \vdots \\ w_7 \\ z_7 \end{bmatrix}
=
\begin{array}{c}
\begin{bmatrix} \ell \\ u-\ell \\ \ell-u \\ u-\ell \\ \ell-u \\ u-\ell \\ \ell-u \\ u-\ell \\ \ell-u \\ u-\ell \\ -u \end{bmatrix}
\end{array}
\begin{array}{c} (b_3) \\ (a_3) \\ (b_4) \\ (a_4) \\ (b_5) \\ (a_5) \\ (b_6) \\ (a_6) \\ (b_7) \\ (a_7) \\ (b_8) \end{array}
$$



Filtering: Sequence

The network
can be analyzed
for filtering in the
same way as the
cardinality
network.



Filtering: genSequence

The genSequence constraint allows arbitrary stretches of variables:

$$\text{genSequence}\,(x \mid \mathcal{X}, V, \ell, u)$$

where

$$\mathcal{X} = (X_1, \ldots, X_m) \quad \ell = (\ell_1, \ldots, \ell_m) \quad u = (u_1, \ldots, u_m)$$

and X_i is a subset of consecutive variables x_i

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It may be possible to permute rows so that the matrix has the consecutive ones property. This allows the network flow model to be used.

This can be checked in $O(m + n + r)$ time, where r = number of nonzeros in matrix.

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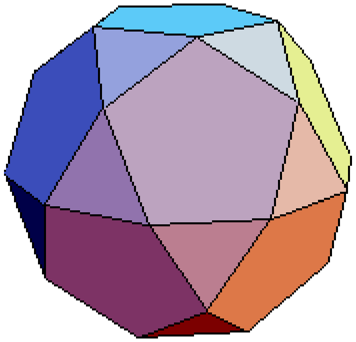
It may be possible to permute rows so that the matrix has the consecutive ones property. This allows the network flow model to be used.

This can be checked in $O(m + n + r)$ time, where r = number of nonzeros in matrix.

Even without consecutive ones, there may be an equivalent network flow matrix. This can be checked in $O(mr)$ time.



Integral Polyhedra

Total Unimodularity

Network Flow Matrices

Interval Matrices

Integral polyhedron

An **integral polyhedron** is one whose vertices have all integral coordinates.

If the continuous relaxation of an MILP model describes an integral polyhedron, the model can be solved as an LP. (All vertices are integral.)

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A matrix is totally unimodular if every square submatrix has determinant 0, 1, or –1.

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A matrix is totally unimodular if every square submatrix has determinant 0, 1, or –1.

Theorem. Matrix A with integral components is totally unimodular if and only if $Ax \geq b$, $x \geq 0$ describes an integral polyhedron for any integral b.

Total unimodularity

Lemma. The following preserve total unimodularity:

- Transposition
- Swapping rows or columns
- Negating a column
- Adding a unit column.

Total unimodularity

Lemma. The following preserve total unimodularity:

- Transposition
- Swapping rows or columns
- Negating a column
- Adding a unit column.

Key Theorem. Matrix A is totally unimodular if and only if every subset J of columns has a partition $J = J_1 \cup J_2$ such that for each row i of A,

$$\left| \sum_{j \in J_1} A_{ij} - \sum_{j \in J_2} A_{ij} \right| \leq 1$$

Total unimodularity

Corollary. A network flow matrix is totally unimodular.

$$
\begin{bmatrix}
1 & 1 & 1 & & & & & \\
-1 & & & 1 & & -1 & & \\
& -1 & & -1 & 1 & & & -1 \\
& & & & -1 & 1 & 1 & \\
& & -1 & & & & -1 & 1
\end{bmatrix}
$$

Total unimodularity

Corollary. A matrix with the consecutive ones property (interval matrix) is totally unimodular.

$$
\begin{bmatrix}
1 & 1 & 1 & & & & & -1 & & & & & & & & & \\
1 & 1 & 1 & & & & & & 1 & & & & & & & & \\
& 1 & 1 & 1 & & & & & & -1 & & & & & & & \\
& 1 & 1 & 1 & & & & & & & 1 & & & & & & \\
& & 1 & 1 & 1 & & & & & & & -1 & & & & & \\
& & 1 & 1 & 1 & & & & & & & & 1 & & & & \\
& & & 1 & 1 & 1 & & & & & & & & -1 & & & \\
& & & 1 & 1 & 1 & & & & & & & & & 1 & & \\
& & & & 1 & 1 & 1 & & & & & & & & & -1 & \\
& & & & 1 & 1 & 1 & & & & & & & & & & 1
\end{bmatrix}
$$

Cutting Planes

0-1 Knapsack Cuts

Gomory Cuts

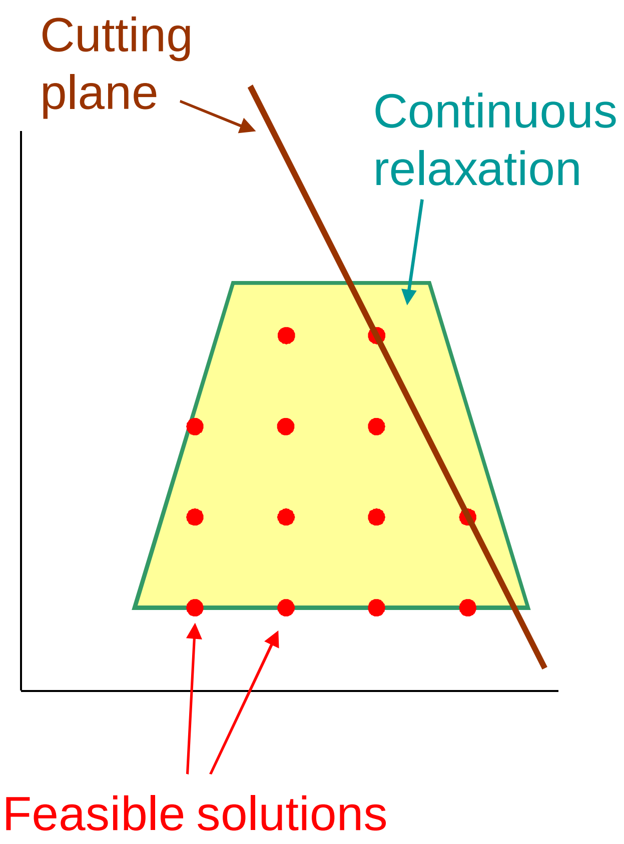
Mixed Integer Rounding Cuts

Example: Product Configuration

To review…

A **cutting plane** (cut, valid inequality) for an MILP model:

- …is **valid**
  - It is satisfied by all feasible solutions of the model.
- …**cuts off** solutions of the continuous relaxation.
  - This makes the relaxation tighter.



Motivation

- **Cutting planes** (cuts) tighten the continuous relaxation of an MILP model.
- **Knapsack cuts**
  - Generated for individual knapsack constraints.
  - We saw **general integer knapsack cuts** earlier.
  - **0-1 knapsack cuts** and **lifting** techniques are well studied and widely used.
- **Rounding cuts**
  - Generated for the entire MILP, they are widely used.
  - **Gomory cuts** for integer variables only.
  - **Mixed integer rounding cuts** for any MILP.

0-1 Knapsack Cuts

0-1 knapsack cuts are designed for knapsack constraints with 0-1 variables.

The analysis is different from that of general knapsack constraints, to exploit the special structure of 0-1 inequalities.

0-1 Knapsack Cuts

0-1 knapsack cuts are designed for knapsack constraints with 0-1 variables.

The analysis is different from that of general knapsack constraints, to exploit the special structure of 0-1 inequalities.

Consider a 0-1 knapsack packing constraint $ax \le a_0$. (Knapsack covering constraints are similarly analyzed.)

Index set J is a **cover** if $\sum_{j \in J} a_j > a_0$

The **cover inequality** $\sum_{j \in J} x_j \le |J| - 1$ is a **0-1 knapsack cut** for $ax \le a_0$

Only **minimal** covers need be considered.

Example

J = {1,2,3,4} is a cover for

$$6x_1 + 5x_2 + 5x_3 + 5x_4 + 8x_5 + 3x_6 \le 17$$

This gives rise to the cover inequality

$$x_1 + x_2 + x_3 + x_4 \le 3$$

Index set J is a **cover** if $\sum_{j \in J} a_j > a_0$

The **cover inequality** $\sum_{j \in J} x_j \le |J| - 1$ is a **0-1 knapsack cut** for $ax \le a_0$

Only **minimal** covers need be considered.

Sequential lifting

- A cover inequality can often be strengthened by **lifting** it into a higher dimensional space.
  - That is, by adding variables.
- **Sequential lifting** adds one variable at a time.
- **Sequence-independent lifting** adds several variables at once.

Sequential lifting

To lift a cover inequality $\sum_{j \in J} x_j \le |J| - 1$

add a term to the left-hand side $\sum_{j \in J} x_j + \pi_k x_k \le |J| - 1$

where π_k is the largest coefficient for which the inequality is still valid.

So, $\pi_k = |J| - 1 - \max_{\substack{x_j \in \{0,1\} \\ \text{for } j \in J}} \left\{ \sum_{j \in J} x_j \,\middle|\, \sum_{j \in J} a_j x_j \le a_0 - a_k \right\}$

This can be done repeatedly (by dynamic programming).

Example

Given $6x_1 + 5x_2 + 5x_3 + 5x_4 + 8x_5 + 3x_6 \le 17$

To lift $x_1 + x_2 + x_3 + x_4 \le 3$

add a term to the left-hand side $x_1 + x_2 + x_3 + x_4 + \pi_5 x_5 \le 3$

where

$$\pi_5 = 3 - \max_{\substack{x_j \in \{0,1\} \\ \text{for } j \in \{1,2,3,4\}}} \left\{ x_1 + x_2 + x_3 + x_4 \,\middle|\, 6x_1 + 5x_2 + 5x_3 + 5x_4 \le 17 - 8 \right\}$$

This yields $x_1 + x_2 + x_3 + x_4 + 2x_5 \le 3$

Further lifting leaves the cut unchanged.

But if the variables are added in the order x_6, x_5, the result is different:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le 3$$

Sequence-independent lifting

- Sequence-independent lifting usually yields a weaker cut than sequential lifting.
    - But it adds all the variables at once and is much faster.
    - Commonly used in commercial MILP solvers.

Sequence-independent lifting

To lift a cover inequality $\sum_{j \in J} x_j \le |J| - 1$

add terms to the left-hand side $\sum_{j \in J} x_j + \sum_{j \notin J} \rho(a_j) x_k \le |J| - 1$

where $\rho(u) = \begin{cases} j & \text{if } A_j \le u \le A_{j+1} - \Delta \text{ and } j \in \{0, \ldots, p-1\} \\ j + (u - A_j)/\Delta & \text{if } A_j - \Delta \le u < A_j - \Delta \text{ and } j \in \{1, \ldots, p-1\} \\ p + (u - A_p)/\Delta & \text{if } A_p - \Delta \le u \end{cases}$

with $\Delta = \sum_{j \in J} a_j - a_0 \qquad A_j = \sum_{k=1}^{j} a_k$

$J = \{1, \ldots, p\} \qquad A_0 = 0$

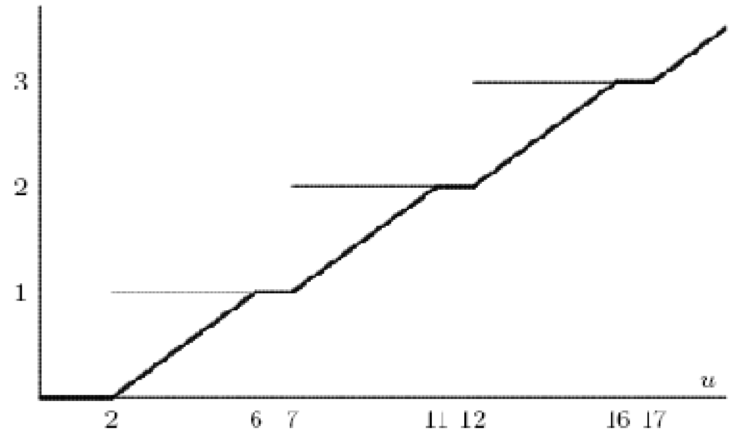
Example

Given $6x_1 + 5x_2 + 5x_3 + 5x_4 + 8x_5 + 3x_6 \leq 17$

To lift $x_1 + x_2 + x_3 + x_4 \leq 3$

Add terms $x_1 + x_2 + x_3 + x_4 + \rho(8)x_5 + \rho(3)x_6 \leq 3$

where $\rho(u)$ is given by



This yields the lifted cut

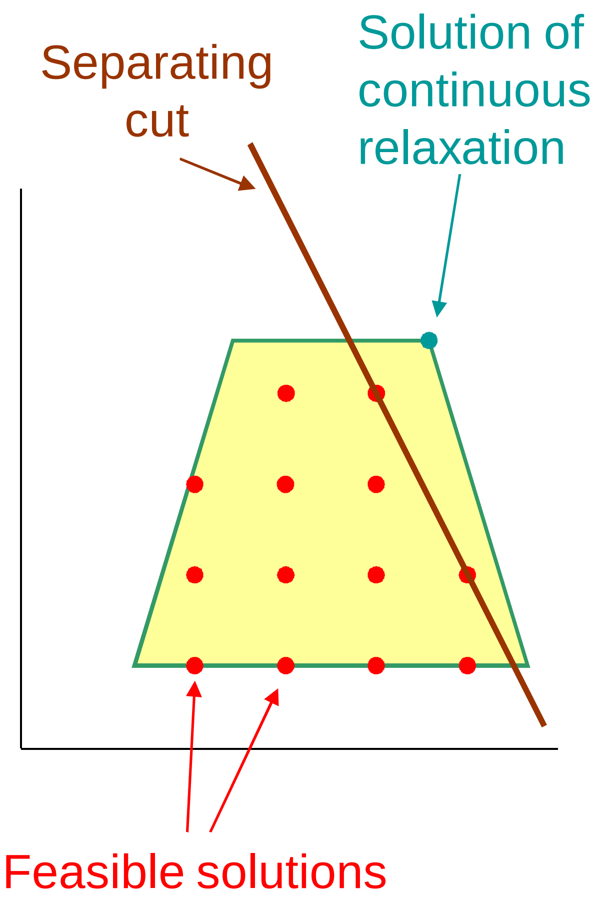
$$x_1 + x_2 + x_3 + x_4 + (5/4)x_5 + (1/4)x_6 \leq 3$$

Gomory Cuts

• When an integer programming problem has a nonintegral solution, we can generate at least one **Gomory cut** to cut off that solution.

- This is a special case of a **separating cut**, because it separates the current solution of the relaxation from the feasible set.

• Gomory cuts are widely used and very effective in MILP solvers.



Gomory cuts

Given an integer programming problem

$$\min cx$$
$$Ax = b$$
$$x \geq 0 \text{ and integral}$$

Let $(x_B, 0)$ be an optimal solution of the continuous relaxation, where

$$x_B = \hat{b} - \hat{N} x_N$$
$$\hat{b} = B^{-1} b, \quad \hat{N} = B^{-1} N$$

Then if x_i is nonintegral in this solution, the following **Gomory cut** is violated by $(x_B, 0)$:

$$x_i + \lfloor \hat{N}_i \rfloor x_N \leq \lfloor \hat{b}_i \rfloor$$

Example

$$\min\ 2x_1 + 3x_2$$
$$x_1 + 3x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1, x_2 \geq 0 \text{ and integral}$$

or

$$\min\ 2x_1 + 3x_2$$
$$x_1 + 3x_2 - x_3 = 3$$
$$4x_1 + 3x_2 - x_4 = 6$$
$$x_j \geq 0 \text{ and integral}$$

Optimal solution of the continuous relaxation has

$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

$$\hat{N} = \begin{bmatrix} 1/3 & -1/3 \\ -4/9 & 1/9 \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

Example

$$\min\ 2x_1 + 3x_2$$
$$x_1 + 3x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
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or

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$$\hat{b} = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

The Gomory cut $x_i + \lfloor \hat{N}_i \rfloor x_N \leq \lfloor \hat{b}_i \rfloor$

is $x_2 + \lfloor [-4/9 \quad 1/9] \rfloor \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \leq \lfloor 2/3 \rfloor$

or $x_2 - x_3 \leq 0$

In x_1,x_2 space this is $x_1 + 2x_2 \geq 3$

Example

$$\min\ 2x_1 + 3x_2$$
$$x_1 + 3x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1, x_2 \geq 0 \text{ and integral}$$

or

$$\min\ 2x_1 + 3x_2$$
$$x_1 + 3x_2 - x_3 = 3$$
$$4x_1 + 3x_2 - x_4 = 6$$
$$x_j \geq 0 \text{ and integral}$$

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$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

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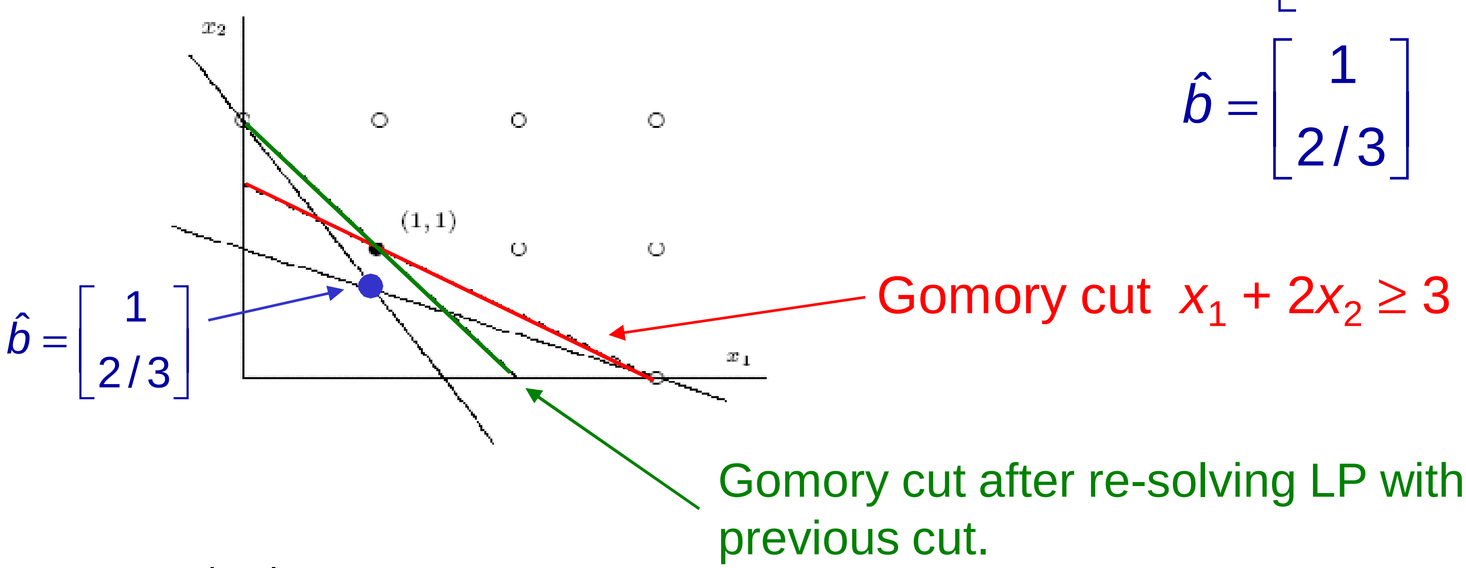


Mixed Integer Rounding Cuts

- **Mixed integer rounding** (MIR) **cuts** can be generated for solutions of any relaxed MILP in which one or more integer variables has a fractional value.
  - Like Gomory cuts, they are separating cuts.
  - MIR cuts are widely used in commercial solvers.

MIR cuts

Given an MILP problem

$$\min cx + dy$$
$$Ax + Dy = b$$
$$x, y \geq 0 \text{ and } y \text{ integral}$$

In an optimal solution of the continuous relaxation, let

$J = \{ j \mid y_j \text{ is nonbasic}\}$

$K = \{ j \mid x_j \text{ is nonbasic}\}$

N = nonbasic cols of $[A\ D]$

Then if y_i is nonintegral in this solution, the following **MIR cut** is violated by the solution of the relaxation:

$$y_i + \sum_{j \in J_1} \lceil \hat{N}_{ij} \rceil y_j + \sum_{j \in J_2} \left(\lfloor \hat{N}_{ij} \rfloor + \frac{\text{frac}(\hat{N}_{ij})}{\text{frac}(\hat{b}_i)} \right) + \frac{1}{\text{frac}(\hat{b}_i)} \sum_{j \in K} \hat{N}_{ij}^{+} x_j \geq \hat{N}_{ij} \lceil \hat{b}_i \rceil$$

where $J_1 = \left\{ j \in J \,\middle|\, \text{frac}(\hat{N}_{ij}) \geq \text{frac}(\hat{b}_j) \right\}$ $\qquad J_2 = J \setminus J_1$

Example

$$3x_1 + 4x_2 - 6y_1 - 4y_2 = 1$$
$$x_1 + 2x_2 - y_1 - y_2 = 3$$
$$x_j, y_j \geq 0, \quad y_j \text{ integer}$$

Take basic solution $(x_1,y_1) = (8/3,17/3)$.

Then $\hat{N} = \begin{bmatrix} 1/3 & 2/3 \\ -2/3 & 8/3 \end{bmatrix}$ $\hat{b} = \begin{bmatrix} 8/3 \\ 17/3 \end{bmatrix}$

$J = \{2\}$, $K = \{2\}$, $J_1 = \varnothing$, $J_2 = \{2\}$

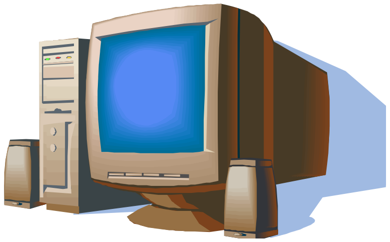
The MIR cut is $y_1 + \left(\lfloor 1/3 \rfloor + \dfrac{1/3}{2/3} \right) y_2 + \dfrac{1}{2/3}(2/3)^+ x_2 \geq \lceil 8/3 \rceil$

or $y_1 + (1/2)y_2 + x_2 \geq 3$

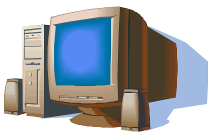
Example: Product Configuration

This example illustrates:

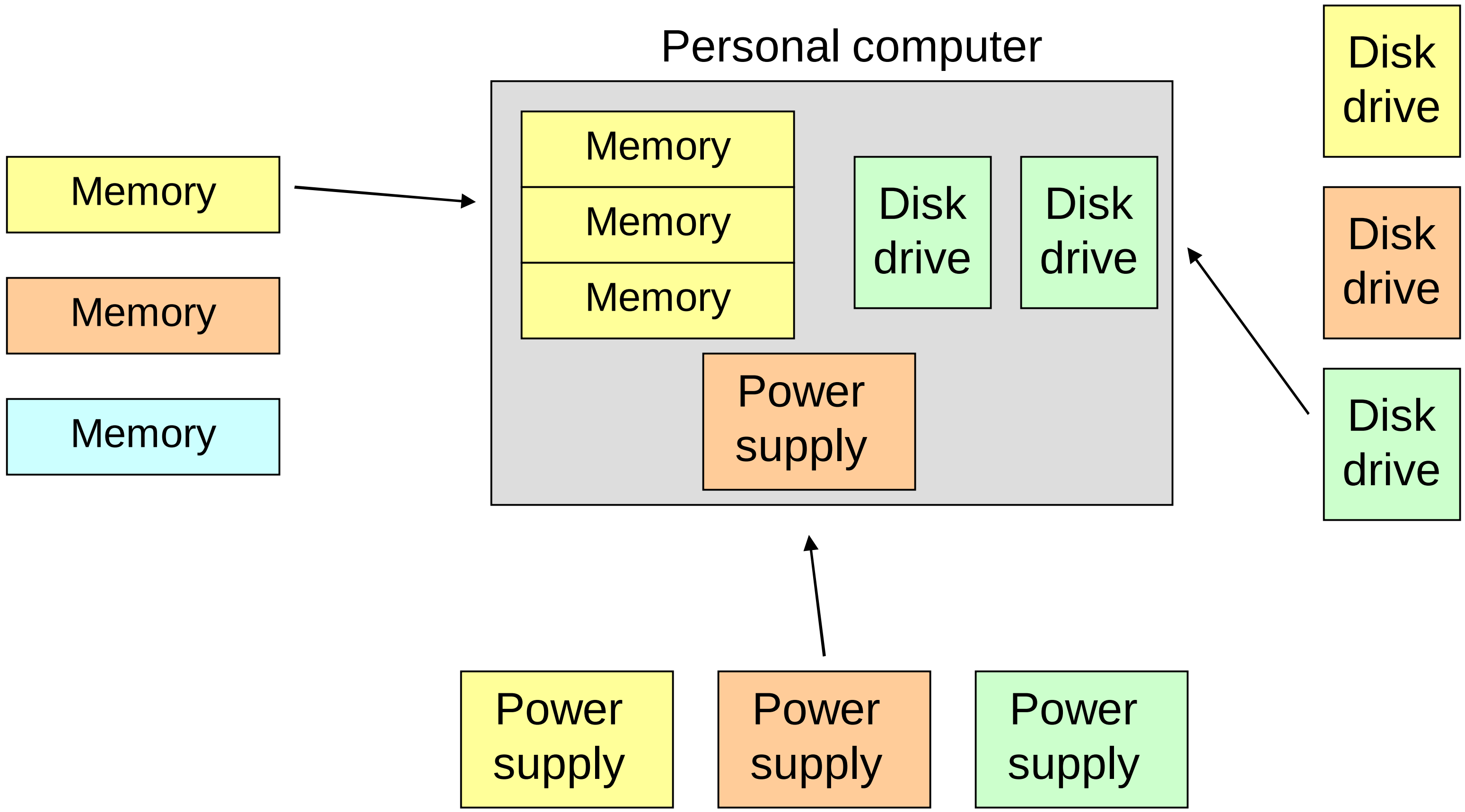
- Combination of propagation and relaxation.
- Processing of variable indices.
- Continuous relaxation of element constraint.



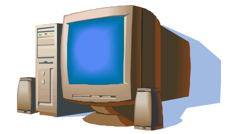
The problem



Choose what type of each component, and how many



Model of the problem



Unit cost of producing attribute *j*

Amount of attribute *j* produced by type t_i of component *i*

$$\min \sum_j c_j v_j$$

Amount of attribute *j* produced (< 0 if consumed): *memory, heat, power, weight, etc.*

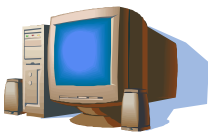
$$v_j = \sum_{ik} q_i A_{ijt_i}, \text{ all } j$$

t_i is a **variable index**

$$L_j \le v_j \le U_j, \text{ all } j$$

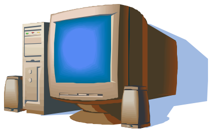
Quantity of component *i* installed

To solve it:



- **Branch** on domains of t_i and q_i.
- **Propagate** *element* constraints and bounds on v_j.
  - **Variable index** is converted to specially structured *element* constraint.
  - Valid **knapsack** cuts are derived and propagated.
- Use linear continuous **relaxations.**
  - Special purpose **MILP** relaxation for *element.*

Propagation



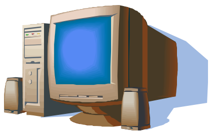
$$\min \sum_j c_j v_j$$

$$v_j = \sum_{ik} q_i A_{ijt_i}, \text{ all } j$$

$$\boxed{L_j \le v_j \le U_j, \text{ all } j}$$

This is propagated in the usual way

Propagation



$$v_j = \sum_i z_i, \text{ all } j$$

$$\text{element}\left(t_i, (q_i A_{ij1}, \ldots, q_i A_{ijn}), z_i\right), \text{ all } i, j$$

$$\min \sum_j c_j v_j$$

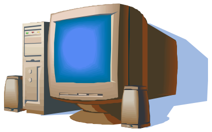
$$\boxed{v_j = \sum_{ik} q_i A_{ijt_i}, \text{ all } j}$$

This is rewritten as

$$\boxed{L_j \le v_j \le U_j, \text{ all } j}$$

This is propagated in the usual way

Propagation

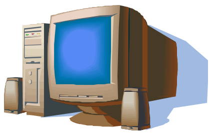


$$v_j = \sum_i z_i,\ \text{all } j$$

$$\text{element}\left(t_i, (q_i A_{ij1}, \ldots, q_i A_{ijn}), z_i\right),\ \text{all } i, j$$

This can be propagated by
(a) using specialized **filters** for *element* constraints of this form…

Propagation



$$v_j = \sum_i z_i, \text{ all } j$$

$$\text{element}\left(t_i, (q_i A_{ij1}, \ldots, q_i A_{ijn}), z_i\right), \text{ all } i, j$$

This is propagated by
(a) using specialized **filters** for *element* constraints of this form,
(b) adding **knapsack cuts** for the valid inequalities:

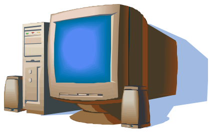
$$\sum_i \max_{k \in D_{t_i}} \{A_{ijk}\} q_i \geq \underline{v}_j, \text{ all } j$$

$$\sum_i \min_{k \in D_{t_i}} \{A_{ijk}\} q_i \leq \bar{v}_j, \text{ all } j$$

and (c) propagating the knapsack cuts.

$[\underline{v}_j, \bar{v}_j]$ is current domain of v_j

Relaxation



$$\min \sum_j c_j v_j$$

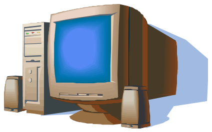
$$v_j = \sum_{ik} q_i A_{ijt_i}, \text{ all } j$$

$$\boxed{L_j \le v_j \le U_j, \text{ all } j}$$

This is relaxed as

$$\underline{v}_j \le v_j \le \bar{v}_j$$

Relaxation



$$v_j = \sum_i z_i, \text{ all } j$$

$$\text{element}\left(t_i, (q_i A_{ij1}, \ldots, q_i A_{ijn}), z_i\right), \text{ all } i, j$$

$$\min \sum_j c_j v_j$$

$$\boxed{v_j = \sum_{ik} q_i A_{ijt_i}, \text{ all } j}$$

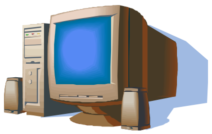
$$\boxed{L_j \le v_j \le U_j, \text{ all } j}$$

This is relaxed by relaxing *this* and adding the knapsack cuts.

This is relaxed as

$$\underline{v}_j \le v_j \le \overline{v}_j$$

Relaxation



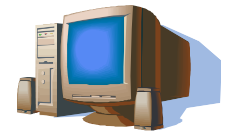
$$v_j = \sum_i z_i, \text{ all } j$$

$$\text{element}\left(t_i, (q_i A_{ij1}, \ldots, q_i A_{ijn}), z_i\right), \text{ all } i, j$$

This is relaxed by replacing each *element* constraint with a disjunctive **convex hull** relaxation:

$$z_i = \sum_{k \in D_{t_i}} A_{ijk} q_{ik}, \quad q_i = \sum_{k \in D_{t_i}} q_{ik}$$

Relaxation



So the following LP relaxation is solved at each node of the search tree to obtain a lower bound:

$$
\begin{aligned}
&\min \ \sum_j c_j v_j \\
&v_j = \sum_i \sum_{k \in D_{t_i}} A_{ijk} q_{ik}, \ \text{all } j \\
&q_j = \sum_{k \in D_{t_i}} q_{ik}, \ \text{all } i \\
&\underline{v}_j \le v_j \le \overline{v}_j, \ \text{all } j \\
&\underline{q}_i \le q_i \le \overline{q}_i, \ \text{all } i \\
&\text{knapsack cuts for } \sum_i \max_{k \in D_{t_i}} \{A_{ijk}\} q_i \ge \underline{v}_j, \ \text{all } j \\
&\text{knapsack cuts for } \sum_i \min_{k \in D_{t_i}} \{A_{ijk}\} q_i \le \overline{v}_j, \ \text{all } j \\
&q_{ik} \ge 0, \ \text{all } i, k
\end{aligned}
$$

Lagrangean Relaxation

Lagrangean Duality
Properties of the Lagrangean Dual
Example: Fast Linear Programming
Domain Filtering
Example: Continuous Global Optimization

Motivation

- **Lagrangean relaxation** can provide better bounds than LP relaxation.
- The **Lagrangean dual** generalizes LP duality.
- It provides **domain filtering** analogous to that based on LP duality.
    - This is a key technique in **continuous global optimization**.
- Lagrangean relaxation gets rid of troublesome constraints by **dualizing** them.
    - That is, moving them into the objective function.
    - The Lagrangean relaxation may **decouple**.

Lagrangean Duality

Consider an
inequality-constrained
problem

$$\min\ f(x)$$

$$g(x) \geq 0$$ Hard constraints

$$x \in S$$ Easy constraints

The object is to get rid of (**dualize**) the hard constraints by moving them into the objective function.

Lagrangean Duality

Consider an inequality-constrained problem

$$\min\ f(x)$$
$$g(x) \geq 0$$
$$x \in S$$

It is related to an inference problem

$$\max\ v$$
$$g(x) \geq b \boxed{\overset{s \in S}{\Longrightarrow}} f(x) \geq v$$

implies

Lagrangean Dual problem: Find the tightest lower bound on the objective function that is implied by the constraints.

Primal

$$\min\ f(x)$$
$$g(x) \geq 0$$
$$x \in S$$

Dual

$$\max\ v$$
$$g(x) \geq b \overset{s \in S}{\Rightarrow} f(x) \geq v$$

Surrogate

Let us say that

$g(x) \geq 0 \overset{x \in S}{\Rightarrow} f(x) \geq v$ iff $\boxed{\lambda g(x) \geq 0}$ $\boxed{\text{dominates}}$ $f(x) - v \geq 0$

for some $\lambda \geq 0$

$\lambda g(x) \leq f(x) - v$ for all $x \in S$

That is, $v \leq f(x) - \lambda g(x)$ for all $x \in S$

Primal

$$\min\ f(x)$$
$$g(x) \geq 0$$
$$x \in S$$

Dual

$$\max\ v$$
$$g(x) \geq b \overset{s \in S}{\Rightarrow} f(x) \geq v$$

Surrogate

Let us say that

$g(x) \geq 0 \overset{x \in S}{\Rightarrow} f(x) \geq v$ iff $\boxed{\lambda g(x) \geq 0}$ $\boxed{\text{dominates}}$ $f(x) - v \geq 0$

for some $\lambda \geq 0$

$\lambda g(x) \leq f(x) - v$ for all $x \in S$

That is, $v \leq f(x) - \lambda g(x)$ for all $x \in S$

Or $v \leq \min_{x \in S} \{f(x) - \lambda g(x)\}$

Primal

$$\min\ f(x)$$
$$g(x) \geq 0$$
$$x \in S$$

Dual

$$\max\ v$$
$$g(x) \geq b \overset{s \in S}{\Rightarrow} f(x) \geq v$$

Surrogate

Let us say that

$g(x) \geq 0 \overset{x \in S}{\Rightarrow} f(x) \geq v$ iff $\boxed{\lambda g(x) \geq 0}$ $\boxed{\text{dominates}}$ $f(x) - v \geq 0$
for some $\lambda \geq 0$

$\lambda g(x) \leq f(x) - v$ for all $x \in S$

That is, $v \leq f(x) - \lambda g(x)$ for all $x \in S$

Or $v \leq \min\limits_{x \in S}\{f(x) - \lambda g(x)\}$

So the dual becomes

$$\max\ v$$
$$v \leq \min_{x \in S}\{f(x) - \lambda g(x)\} \text{ for some } \lambda \geq 0$$

Now we have…

Primal

$$\min\ f(x)$$
$$\boxed{g(x) \geq 0}$$
$$x \in S$$

These constraints are **dualized**

Dual

$$\max\ v$$
$$v \leq \min_{x \in S}\{f(x) - \lambda g(x)\} \text{ for some } \lambda \geq 0$$

or

$$\max_{\lambda \geq 0} \theta(\lambda)$$

where

$$\theta(\lambda) = \min_{x \in S}\{f(x) - \lambda g(x)\}$$

Lagrangean relaxation

Vector of **Lagrange multipliers**

The Lagrangean dual can be viewed as the problem of finding the Lagrangean relaxation that gives the tightest bound.

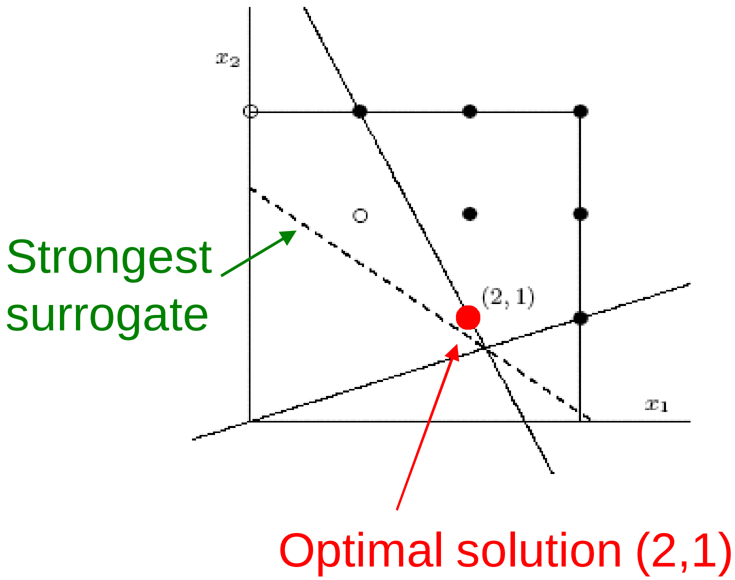
Example

$$\begin{aligned}&\min\ 3x_1+4x_2\\&-x_1+3x_2\ge 0\\&2x_1+x_2-5\ge 0\\&x_1,x_2\in\{0,1,2,3\}\end{aligned}$$

The Lagrangean relaxation is

$$\theta(\lambda_1,\lambda_2)=\min_{x_j\in\{0,\ldots,3\}}\{3x_1+4x_2-\lambda_1(-x_1+3x_2)-\lambda_2(2x_1+x_2-5)\}$$

$$=\min_{x_j\in\{0,\ldots,3\}}\{(3+\lambda_1-2\lambda_2)x_1+(4-3\lambda_1-\lambda_2)x_2+5\lambda_2\}$$



The Lagrangean relaxation is easy to solve for any given λ_1, λ_2:

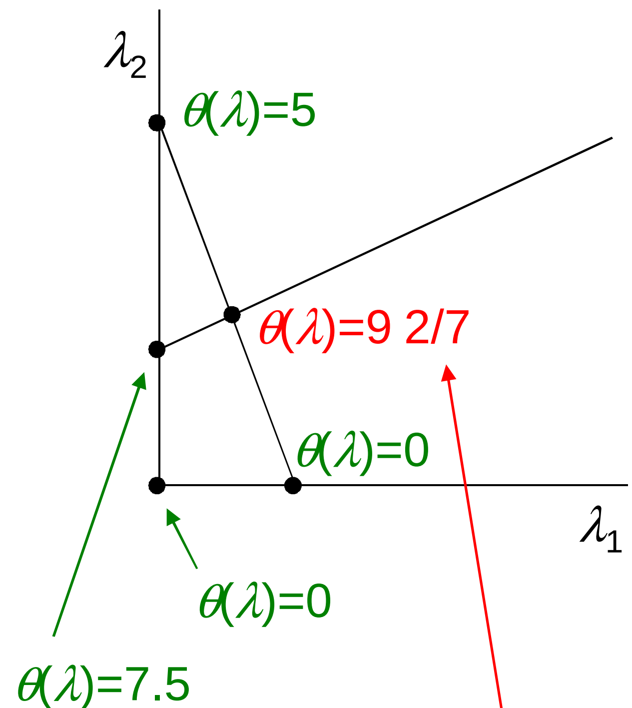
$$x_1=\begin{cases}0 & \text{if } 3+\lambda_1-2\lambda_2\ge 0\\ 3 & \text{otherwise}\end{cases}$$

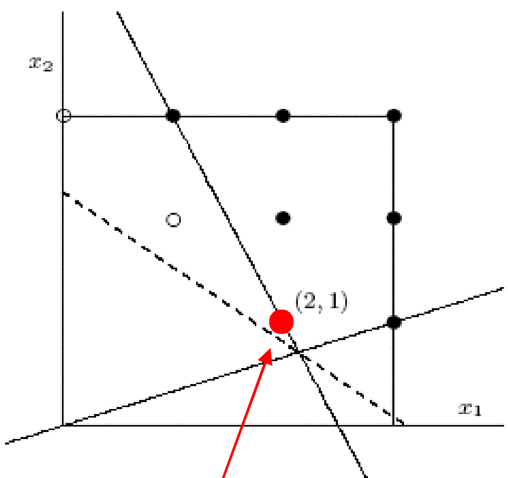
$$x_2=\begin{cases}0 & \text{if } 4-3\lambda_1-\lambda_2\ge 0\\ 3 & \text{otherwise}\end{cases}$$

Example

$\theta(\lambda_1,\lambda_2)$ is piecewise linear and concave.

$$\begin{aligned}&\min\ 3x_1 + 4x_2\\ &-x_1 + 3x_2 \geq 0\\ &2x_1 + x_2 - 5 \geq 0\\ &x_1, x_2 \in \{0,1,2,3\}\end{aligned}$$





Solution of Lagrangean dual:

(λ_1,λ_2) = (5/7, 13/7), $\theta(\lambda)$ = 9 2/7

Optimal solution (2,1)
Value = 10

Note **duality gap** between 10 and 9 2/7 (no strong duality).

Example

$$\min\ 3x_1 + 4x_2$$
$$-x_1 + 3x_2 \geq 0$$
$$2x_1 + x_2 - 5 \geq 0$$
$$x_1, x_2 \in \{0,1,2,3\}$$

Note: in this example, the Lagrangean dual provides the same bound (9 2/7) as the continuous relaxation of the IP.

This is because the Lagrangean relaxation can be solved as an LP:

$$\theta(\lambda_1,\lambda_2) = \min_{x_j \in \{0,\ldots,3\}} \left\{(3+\lambda_1-2\lambda_2)x_1 + (4-3\lambda_1-\lambda_2)x_2 + 5\lambda_2\right\}$$
$$= \min_{0 \leq x_j \leq 3} \left\{(3+\lambda_1-2\lambda_2)x_1 + (4-3\lambda_1-\lambda_2)x_2 + 5\lambda_2\right\}$$

Lagrangean duality is useful when the Lagrangean relaxation is tighter than an LP but nonetheless easy to solve.

Properties of the Lagrangean dual

Weak duality: For any feasible x^* and any $\lambda^* \geq 0$, $f(x^*) \geq \theta(\lambda^*)$.

In particular,

$$\begin{array}{l} \min\ f(x) \\ g(x) \geq 0 \\ x \in S \end{array} \quad \geq \quad \max_{\lambda \geq 0} \theta(\lambda)$$

Concavity: $\theta(\lambda)$ is concave. It can therefore be maximized by local search methods.

Complementary slackness: If x^* and λ^* are optimal, and there is no duality gap, then $\lambda^* g(x^*) = 0$.

Solving the Lagrangean dual

Let λ^k be the kth iterate, and let $\lambda^{k+1} = \lambda^k + \alpha_k \boxed{\xi^k}$

Subgradient of $\theta(\lambda)$ at $\lambda = \lambda^k$

If x^k solves the Lagrangean relaxation for $\lambda = \lambda^k$, then $\xi^k = g(x^k)$.

This is because $\theta(\lambda) = f(x^k) + \lambda g(x^k)$ at $\lambda = \lambda^k$.

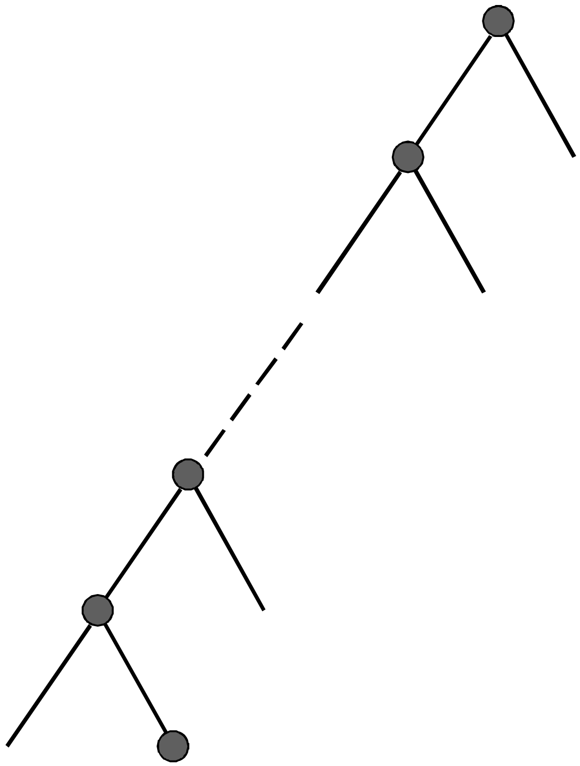
The stepsize α_k must be adjusted so that the sequence converges but not before reaching a maximum.

Example: Fast Linear Programming

• In CP contexts, it is best to process each node of the search tree very rapidly.

• Lagrangean relaxation may allow very fast calculation of a lower bound on the optimal value of the LP relaxation at each node.

• The idea is to solve the Lagrangean dual at the root node (which is an LP) and use the same Lagrange multipliers to get an LP bound at other nodes.



At root node, solve

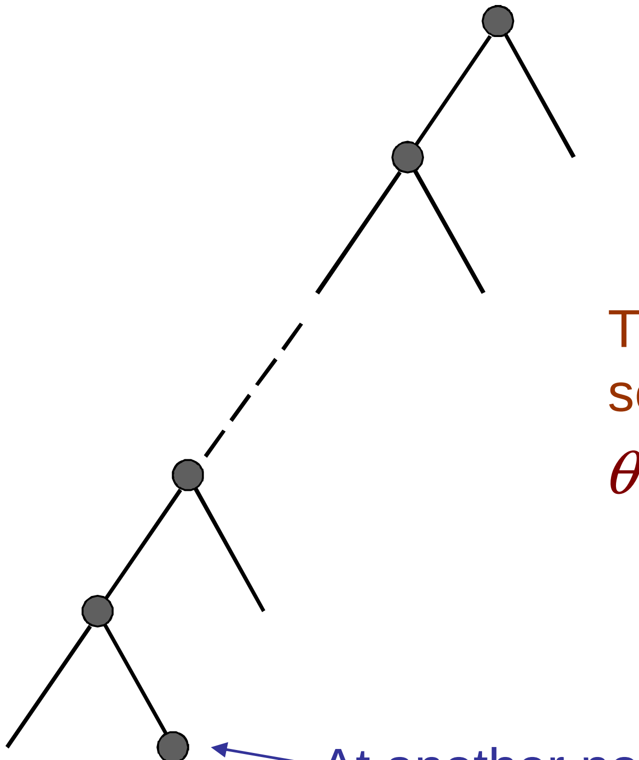
$$
\begin{aligned}
\min\ & cx \\
& Ax \ge b \quad (\lambda) \\
& Dx \ge d \\
& x \ge 0
\end{aligned}
$$

Dualize

Special structure,
e.g. variable bounds

The (partial) LP dual solution λ^* solves the Lagrangean dual in which

$$\theta(\lambda) = \min_{\substack{Dx \ge d \\ x \ge 0}} \{cx - \lambda(Ax - b)\}$$



At root node, solve

$$\begin{aligned} \min\ & cx \\ & Ax \geq b \quad (\lambda) \\ & Dx \geq d \\ & x \geq 0 \end{aligned}$$

Dualize

Special structure, e.g. variable bounds

The (partial) LP dual solution λ^* solves the Lagrangean dual in which

$$\theta(\lambda) = \min_{\substack{Dx \geq d \\ x \geq 0}} \{ cx - \lambda(Ax - b) \}$$

At another node, the LP is

$$\begin{aligned} \min\ & cx \\ & Ax \geq b \quad (\lambda) \\ & Dx \geq d \\ & Hx \geq h \\ & x \geq 0 \end{aligned}$$

Branching constraints, etc.

Here $\theta(\lambda^*)$ is still a lower bound on the optimal value of the LP and can be quickly calculated by solving a specially structured LP.

Domain Filtering

Suppose:

$$\begin{aligned} &\min\ f(x) \\ &g(x) \geq 0 \\ &x \in S \end{aligned}$$

has optimal solution x^*, optimal value v^*, and optimal Lagrangean dual solution λ^*.

…and $\lambda_i^* > 0$, which means the i-th constraint is tight (complementary slackness);

…and the problem is a relaxation of a CP problem;

…and we have a feasible solution of the CP problem with value U, so that U is an upper bound on the optimal value.

Supposing $\begin{array}{l} \min \; f(x) \\ g(x) \geq 0 \\ x \in S \end{array}$ has optimal solution x^*, optimal value v^*, and optimal Lagrangean dual solution λ^*:

If x were to change to a value other than x^*, the LHS of i-th constraint $g_i(x) \geq 0$ would change by some amount Δ_i.

Since the constraint is tight, this would increase the optimal value as much as changing the constraint to $g_i(x) - \Delta_i \geq 0$.

So it would increase the optimal value at least $\lambda_i^* \Delta_i$.

(It is easily shown that Lagrange multipliers are marginal costs. Dual multipliers for LP are a special case of Lagrange multipliers.)

Supposing $\begin{array}{l} \min\ f(x) \\ g(x) \geq 0 \\ x \in S \end{array}$ has optimal solution x^*, optimal value v^*, and optimal Lagrangean dual solution λ^*:

We have found: a change in x that changes $g_i(x)$ by Δ_i increases the optimal value at least $\lambda_i^* \Delta_i$.

Since optimal value of this problem $\leq$ optimal value of the CP $\leq U$, we have $\lambda_i^* \Delta_i \leq U - v^*$, or

$$\Delta_i \leq \frac{U - v^*}{\lambda_i^*}$$

Supposing $\begin{array}{l} \min\ f(x) \\ g(x) \geq 0 \\ x \in S \end{array}$ has optimal solution x^*, optimal value v^*, and optimal Lagrangean dual solution λ^*:

We have found: a change in x that changes $g_i(x)$ by Δ_i increases the optimal value at least $\lambda_i^* \Delta_i$.

Since optimal value of this problem $\leq$ optimal value of the CP $\leq U$,
we have $\lambda_i^* \Delta_i \leq U - v^*$, or

$$\Delta_i \leq \frac{U - v^*}{\lambda_i^*}$$

Since $\Delta_i = g_i(x) - g_i(x^*) = g_i(x)$, this implies the inequality

$$g_i(x) \leq \frac{U - v^*}{\lambda_i^*}$$

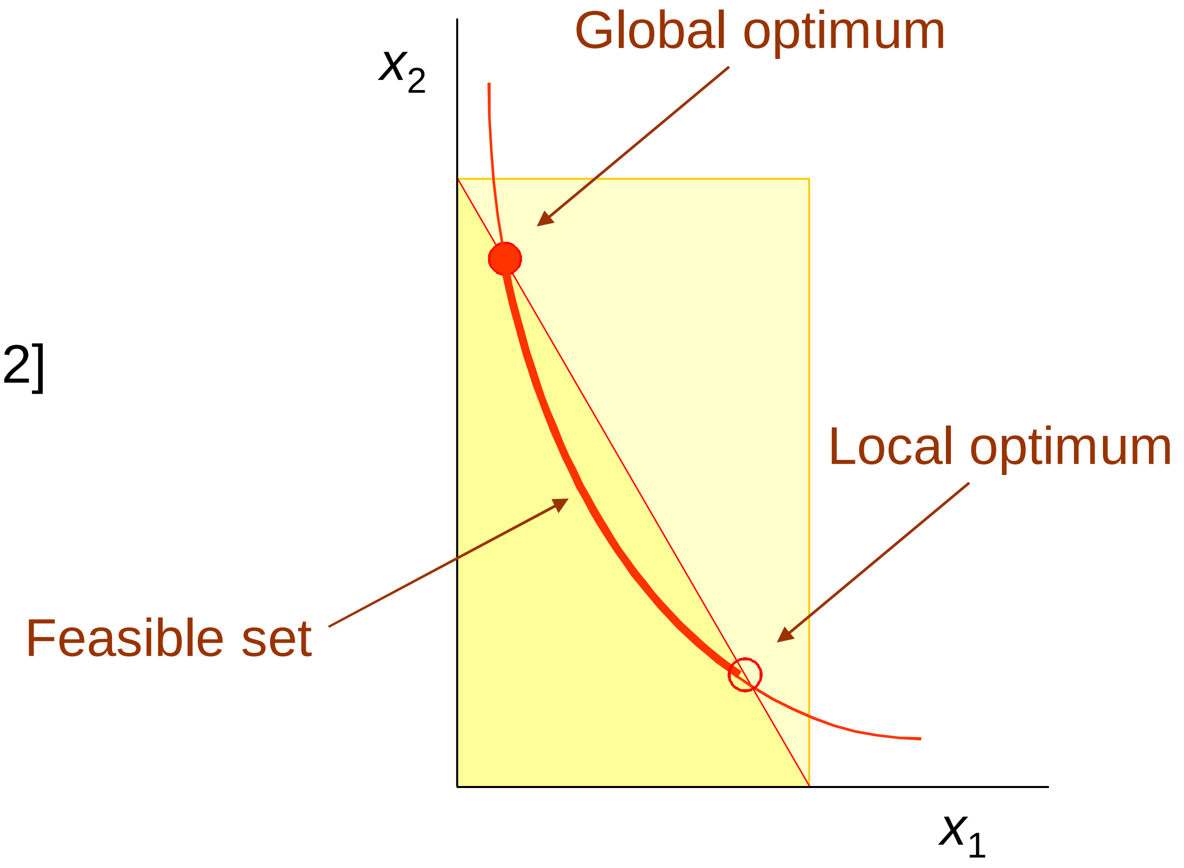
…which can be propagated.

Example: Continuous Global Optimization

- Some of the best continuous global solvers (e.g., BARON) combine OR-style relaxation with CP-style interval arithmetic and domain filtering.
- These methods can be combined with domain filtering based on Lagrange multipliers.

Continuous Global Optimization

$$\max \ x_1 + x_2$$
$$4x_1x_2 = 1$$
$$2x_1 + x_2 \le 2$$
$$x_1 \in [0,1], \quad x_2 \in [0,2]$$

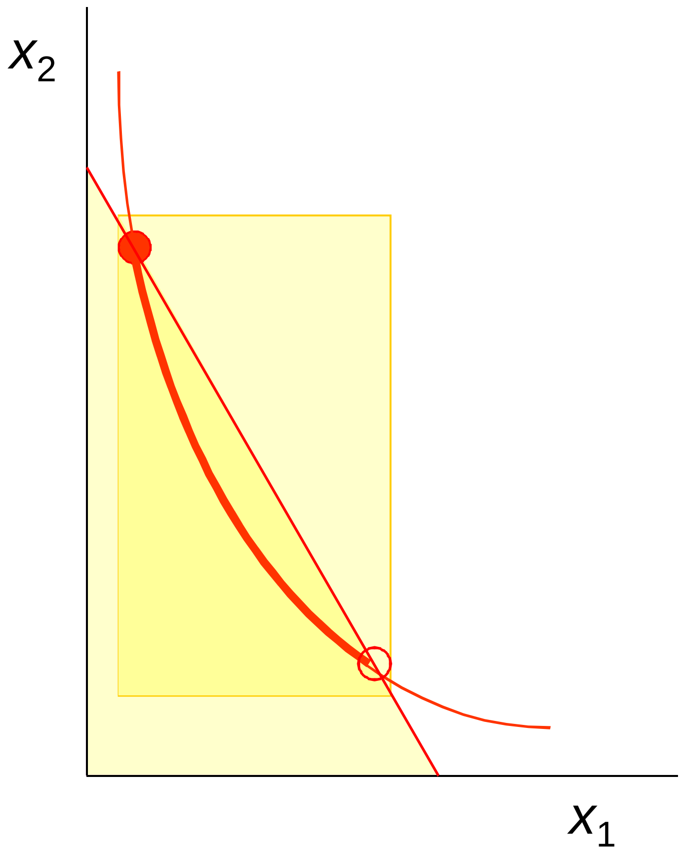


To solve it:

- **Search**: split interval domains of x_1, x_2.
  - Each **node** of search tree is a problem restriction.
- **Propagation:** Interval propagation, domain filtering.
  - Use **Lagrange multipliers** to infer valid inequality for propagation.
  - **Reduced-cost variable** fixing is a special case.
- **Relaxation:** Use function **factorization** to obtain linear continuous relaxation.

Interval propagation

Propagate intervals
[0,1], [0,2]
through constraints
to obtain
[1/8,7/8], [1/4,7/4]



Relaxation (function factorization)

Factor complex functions into elementary functions that have known linear relaxations.

Write $4x_1x_2 = 1$ as $4y = 1$ where $y = x_1x_2$.

This factors $4x_1x_2$ into linear function $4y$ and bilinear function x_1x_2.

Linear function $4y$ is its own linear relaxation.

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This factors $4x_1x_2$ into linear function $4y$ and bilinear function x_1x_2.

Linear function $4y$ is its own linear relaxation.

Bilinear function $y = x_1x_2$ has relaxation:

$$\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1\underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1\underline{x}_2$$
$$\bar{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1\bar{x}_2 \le y \le \bar{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1\bar{x}_2$$

where domain of x_j is $[\underline{x}_j, \bar{x}_j]$

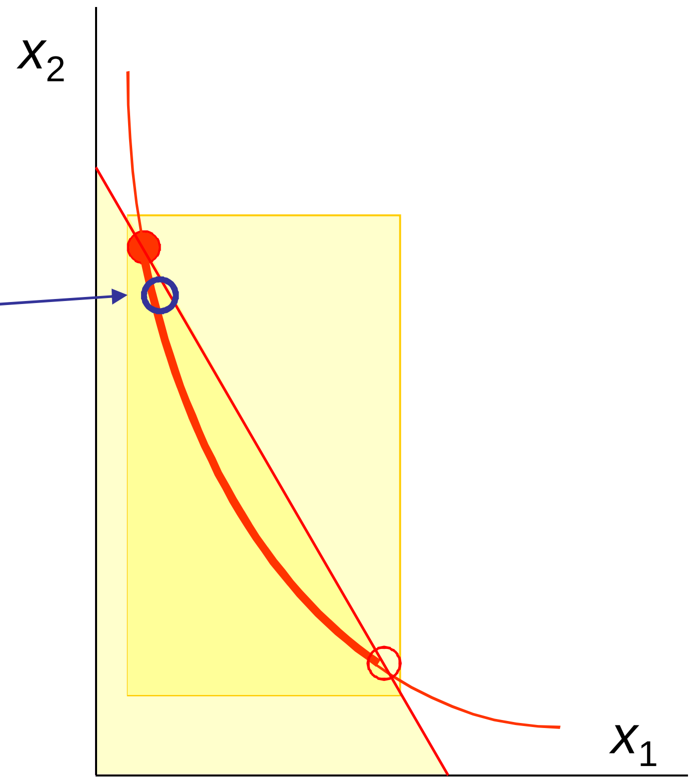
Relaxation (function factorization)

The linear relaxation becomes:

$$
\begin{aligned}
&\min \; x_1 + x_2 \\
&4y = 1 \\
&2x_1 + x_2 \le 2 \\
&\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \overline{x}_1 x_2 - \overline{x}_1 \underline{x}_2 \\
&\overline{x}_2 x_1 + \overline{x}_1 x_2 - \overline{x}_1 \overline{x}_2 \le y \le \overline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \overline{x}_2 \\
&\underline{x}_j \le x_j \le \overline{x}_j, \quad j = 1,2
\end{aligned}
$$

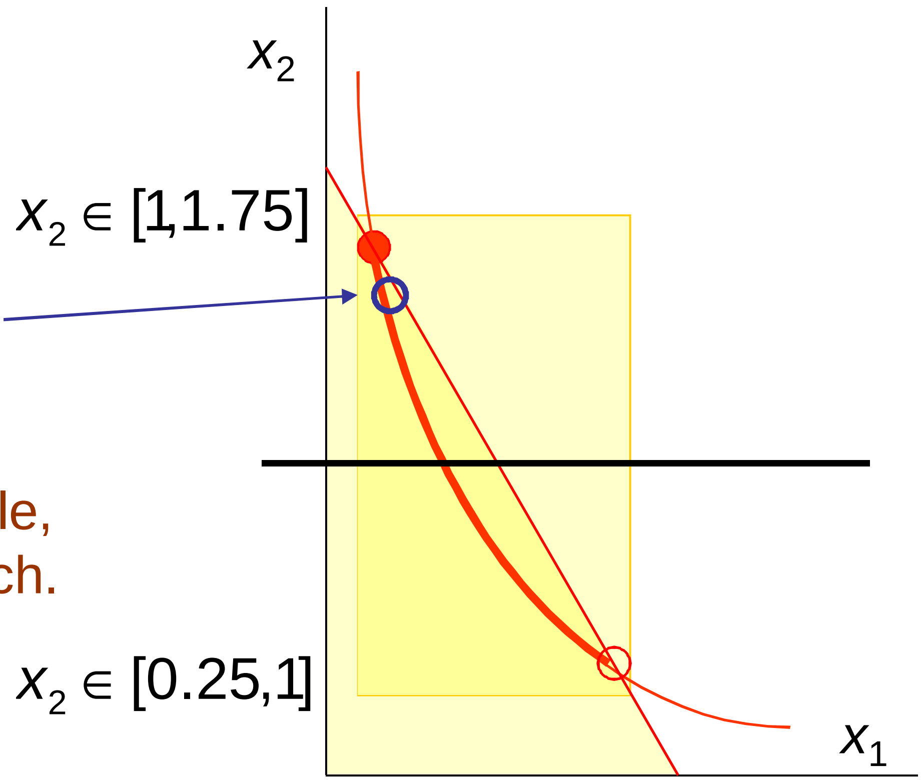
Relaxation (function factorization)

Solve linear relaxation.

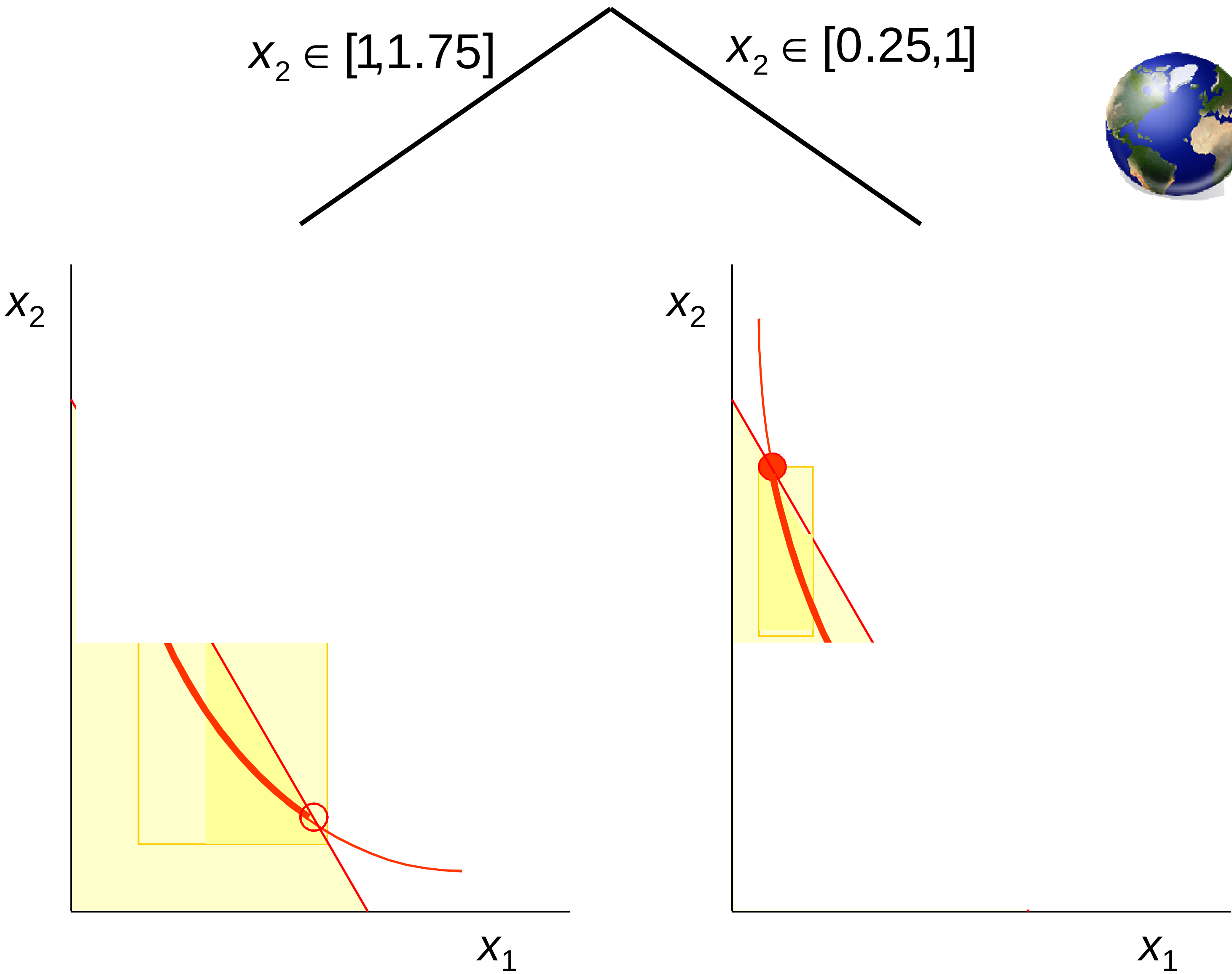


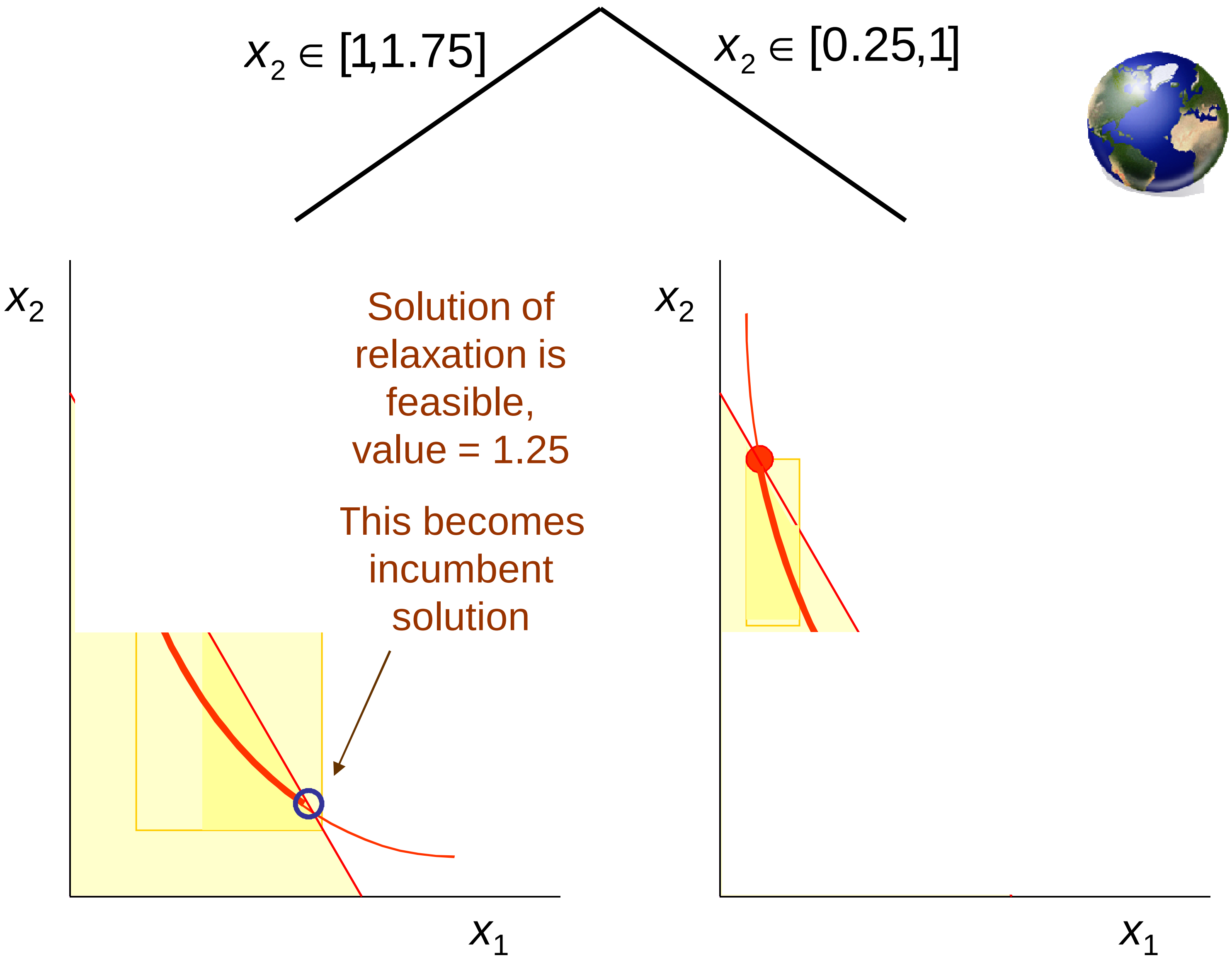
Relaxation (function factorization)

Solve linear relaxation.

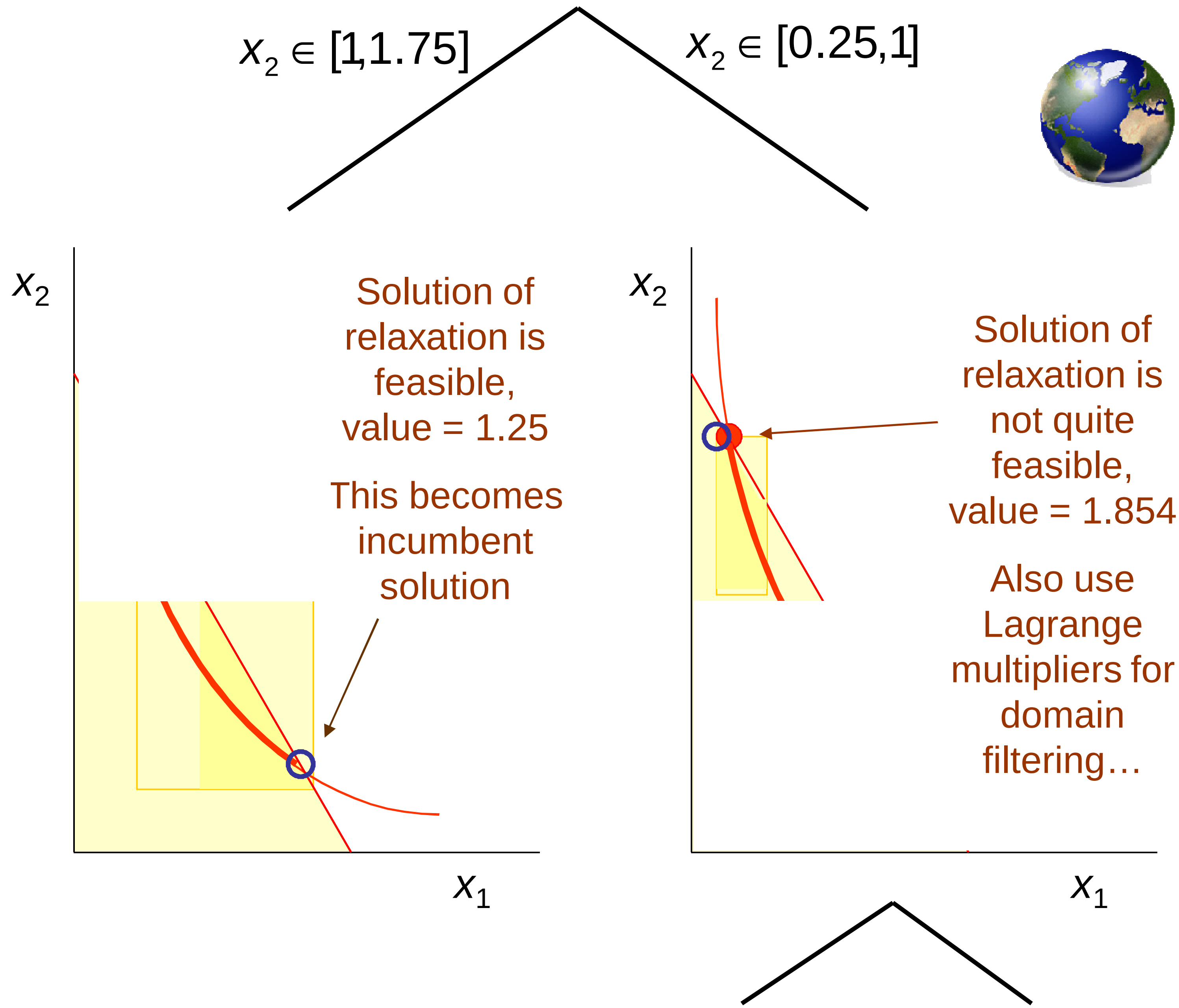


Since solution is infeasible,
split an interval and branch.


$x_2 \in [1,1.75]$
$x_2 \in [0.25,1]$
x_2
x_1
x_2
x_1



$x_2 \in [1,1.75]$
$x_2 \in [0.25,1]$
Solution of relaxation is feasible, value = 1.25
This becomes incumbent solution
x_2
x_1
x_2
x_1


$x_2 \in [1,1.75]$
$x_2 \in [0.25,1]$
x_2
x_1
Solution of relaxation is feasible, value = 1.25
This becomes incumbent solution
x_2
x_1
Solution of relaxation is not quite feasible, value = 1.854
Also use Lagrange multipliers for domain filtering…

Relaxation (function factorization)

$$\min \; x_1 + x_2$$
$$4y = 1$$
$$2x_1 + x_2 \le 2$$
$$\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \underline{x}_2$$
$$\bar{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \bar{x}_2 \le y \le \bar{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \bar{x}_2$$
$$\underline{x}_j \le x_j \le \bar{x}_j, \quad j = 1,2$$

Associated Lagrange multiplier in solution of relaxation is $\lambda_2 = 1.1$

Relaxation (function factorization)

$$\min\ x_1 + x_2$$
$$4y = 1$$
$$2x_1 + x_2 \le 2$$
$$\underline{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \underline{x}_2 \le y \le \underline{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \underline{x}_2$$
$$\bar{x}_2 x_1 + \bar{x}_1 x_2 - \bar{x}_1 \bar{x}_2 \le y \le \bar{x}_2 x_1 + \underline{x}_1 x_2 - \underline{x}_1 \bar{x}_2$$
$$\underline{x}_j \le x_j \le \bar{x}_j,\quad j = 1,2$$

Associated Lagrange multiplier in solution of relaxation is $\lambda_2 = 1.1$

This yields a valid inequality for propagation:

$$2x_1 + x_2 \ge 2 - \frac{\boxed{1.854} - \boxed{1.25}}{\boxed{1.1}} = 1.451$$

Value of relaxation

Lagrange multiplier

Value of incumbent solution

Dynamic Programming in CP

Example: Capital Budgeting

Domain Filtering

Recursive Optimization

Filtering for Stretch

Filtering for Regular

Motivation

- **Dynamic programming** (DP) is a highly versatile technique that can exploit recursive structure in a problem.
- **Domain filtering** is straightforward for problems modeled as a DP.
- DP is also important in designing **filters** for some global constraints, such as *stretch* and *regular*.
- **Nonserial DP** is related to bucket elimination in CP and exploits the structure of the primal graph.
- DP modeling is the **art** of keeping the state space small while maintaining a Markovian property.
- We will examine only **one simple example** of serial DP.

Example: Capital Budgeting

We wish to built power plants with a total cost of at most 12 million Euros.

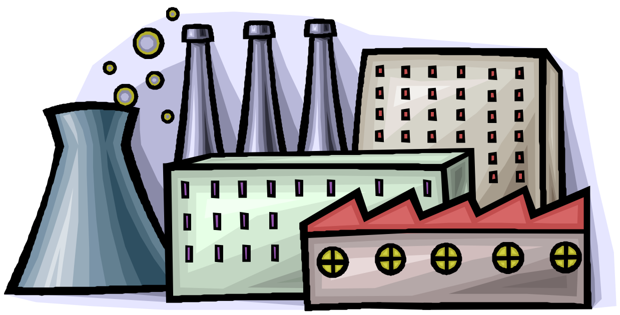
There are three types of plants, costing 4, 2 or 3 million Euros each. We must build one or two of each type.

The problem has a simple knapsack packing model:

$$4x_1 + 2x_2 + 3x_3 \leq 12$$

Number of factories of type *j*

$$\boxed{x_j} \in \{1,2\}$$



Example: Capital Budgeting

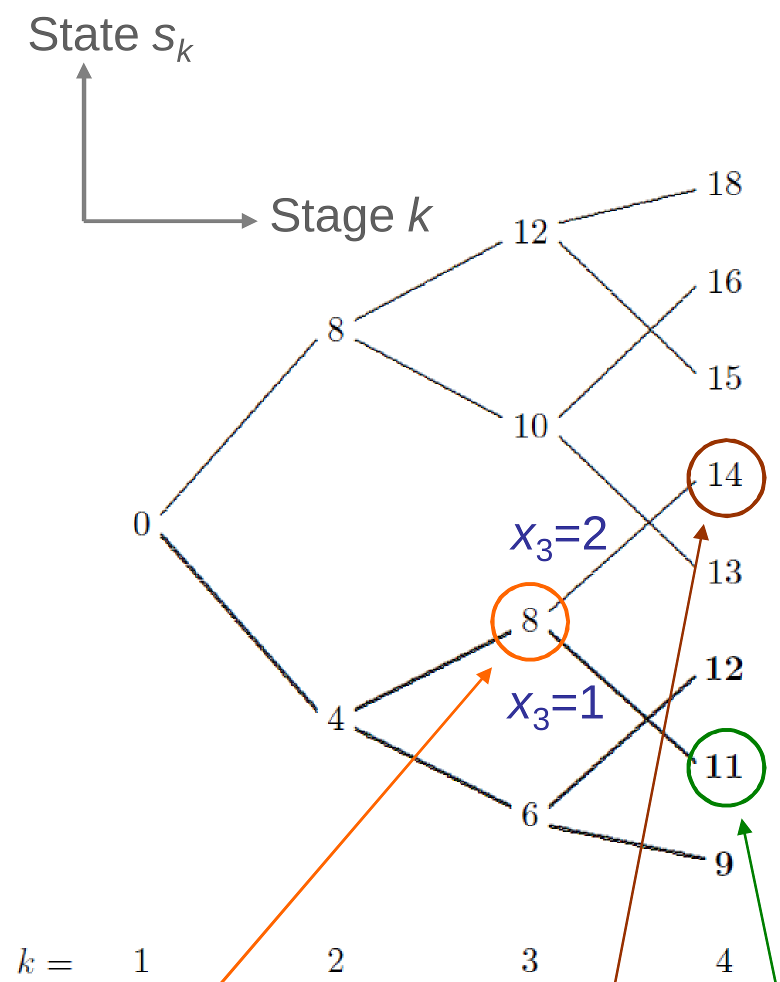
$$4x_1 + 2x_2 + 3x_3 \le 12$$
$$x_j \in \{1,2\}$$

The recursion for $ax \le b$ might be

$$f_k(s_k) = \sum_{x_k \in D_{x_k}} f_{k+1}(s_k + a_k x_k)$$

= # of paths from state s_k to feasible solutions

State is sum of first k terms of ax



$f_3(8) = f_4(8+3\cdot1) + f_4(8+3\cdot2)\} = 1 + 0 = 1$

$f_4(14)=0$

$f_4(11)=1$

Example: Capital Budgeting

$$4x_1 + 2x_2 + 3x_3 \leq 12$$
$$x_j \in \{1,2\}$$

The recursion for $ax \leq b$ might be

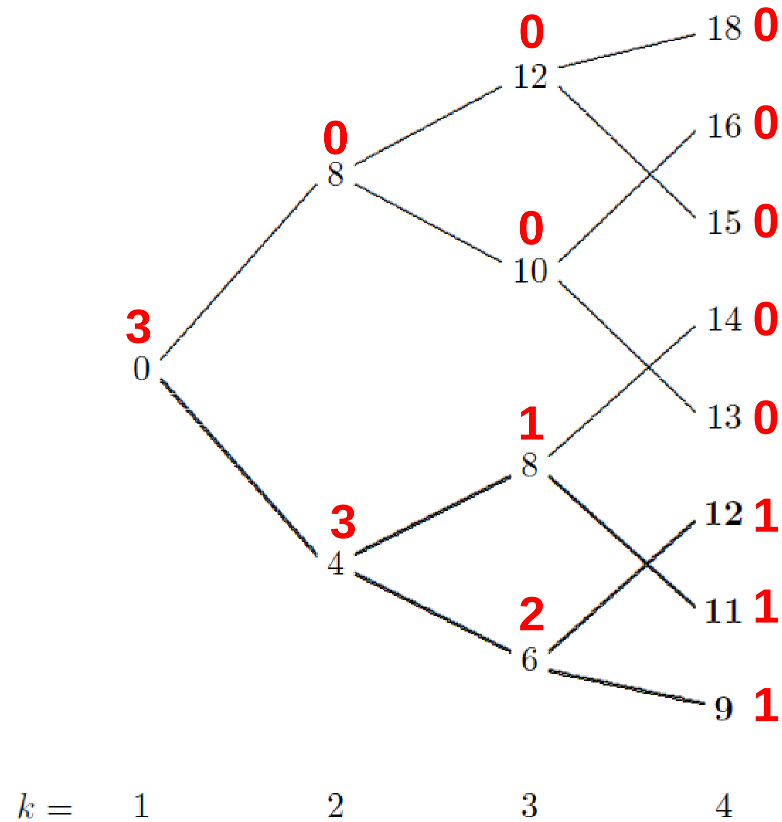
$$f_k(s_k) = \sum_{x_k \in D_{x_k}} f_{k+1}(s_k + a_k x_k)$$

Boundary condition:

$$f_{n+1}(s_{n+1}) = \begin{cases} 1 & \text{if } s_{n+1} \leq b \\ 0 & \text{otherwise} \end{cases}$$

Feasible if:

$$f_1(0) > 0$$



$f_k(s_k)$ for each state s_k

Example: Capital Budgeting

$$4x_1 + 2x_2 + 3x_3 \le 12$$
$$x_j \in \{1,2\}$$

The problem is feasible.

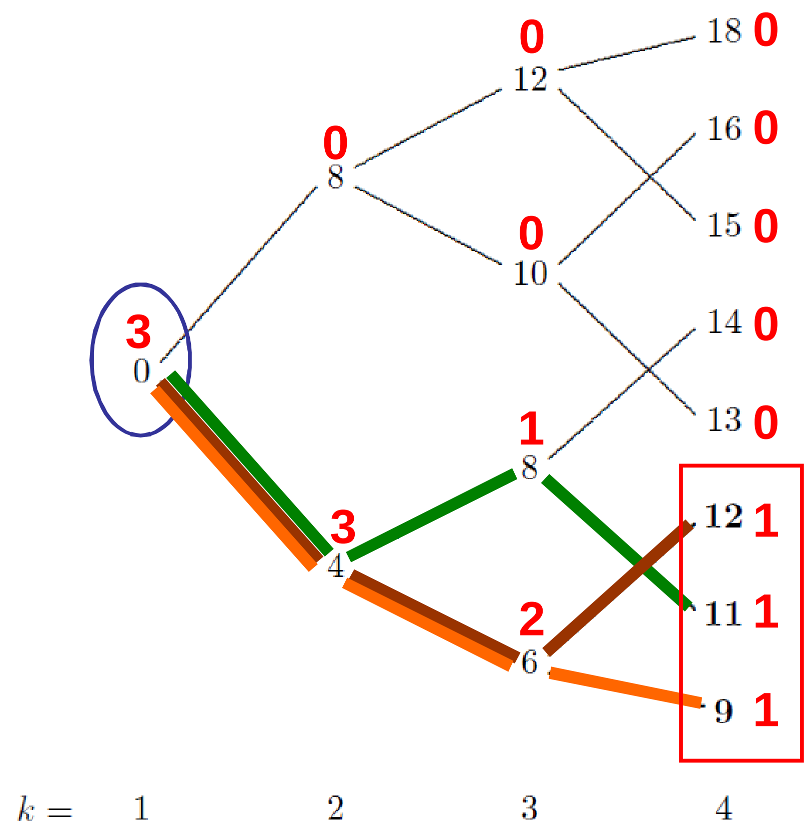
Each path to 1 is a feasible solution.

Path 1: x = (1,2,1)

Path 2: x = (1,1,2)

Path 3: x = (1,1,1)

Possible costs are 9,11,12.



$f_k(s_k)$ for each state s_k

Example: Capital Budgeting

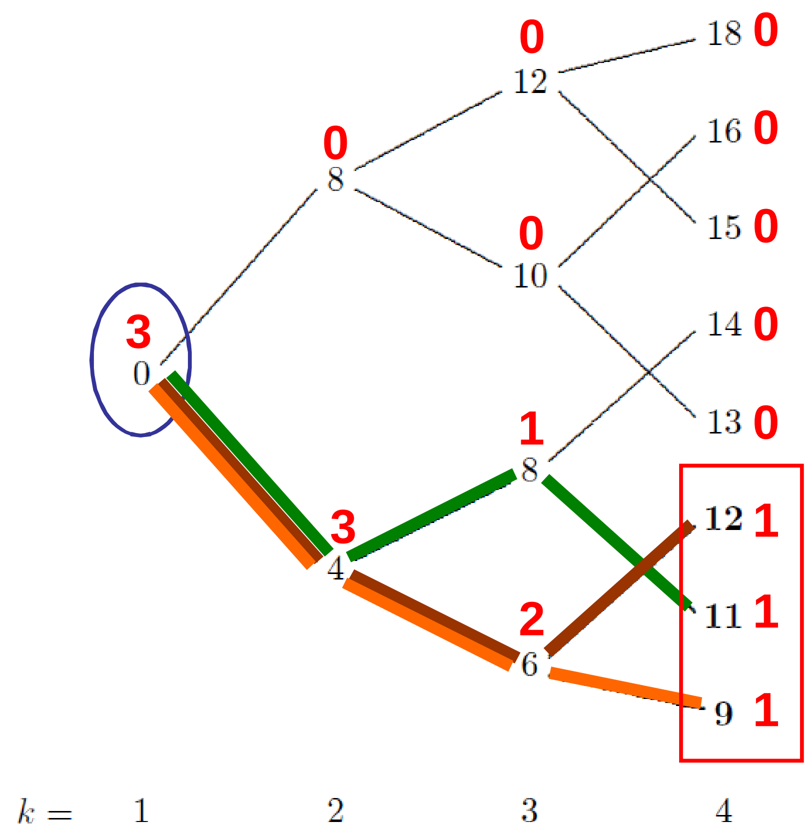
$$4x_1 + 2x_2 + 3x_3 \leq 12$$
$$x_j \in \{1,2\}$$

Key property:

The DP model is **Markovian**

Possible transitions depend only on current state…

…not how the state was reached.



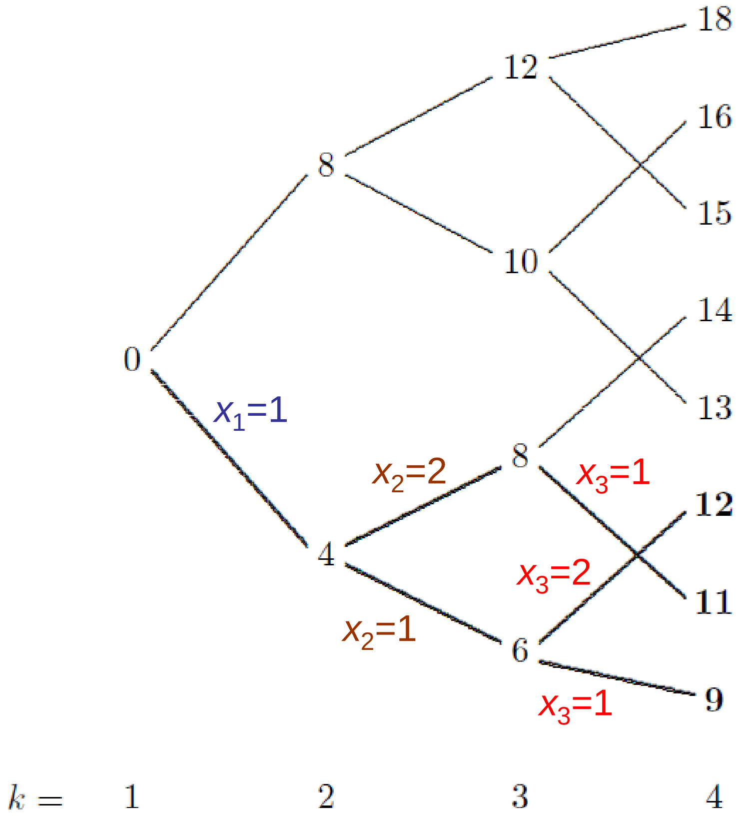
$f_k(s_k)$ for each state s_k

Domain Filtering

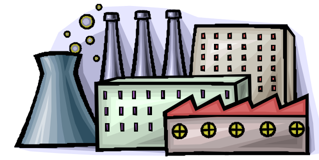
$$4x_1 + 2x_2 + 3x_3 \leq 12$$
$$x_j \in \{1,2\}$$

To filter domains: observe what values of x_k occur on feasible paths.

$$D_{x_3} = \{1,2\}$$
$$D_{x_2} = \{1,2\}$$
$$D_{x_1} = \{1\}$$



Recursive Optimization



$$\max\ 15x_1 + 10x_2 + 12x_3$$
$$4x_1 + 2x_2 + 3x_3 \le 12$$
$$x_j \in \{1, 2\}$$

Maximize revenue

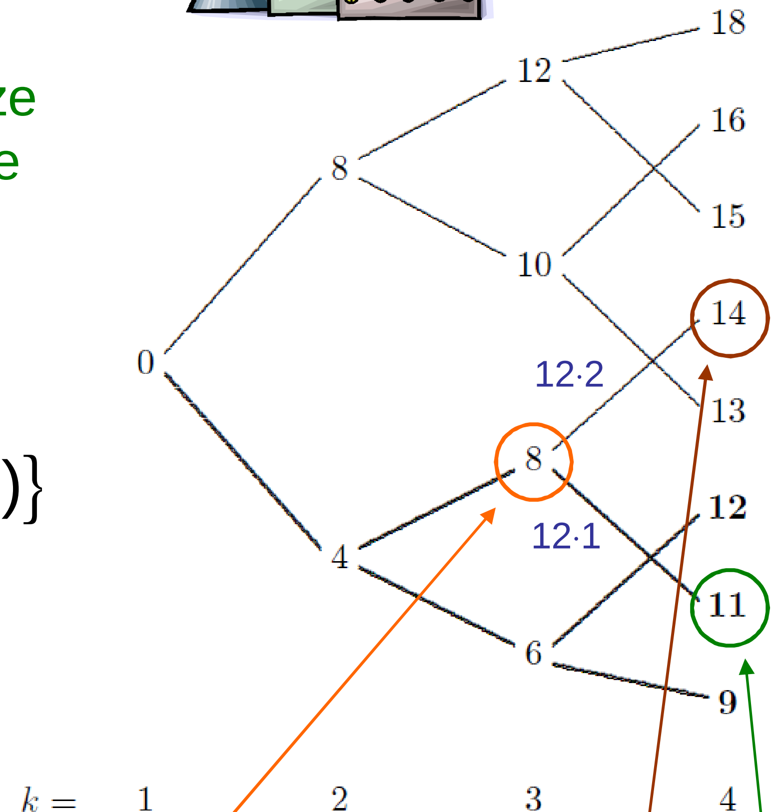
The recursion includes arc values:

$$f_k(s_k) = \max_{x_k \in D_{x_k}} \{c_k x_k + f_{k+1}(s_k + a_k x_k)\}$$

= value on max value path from s_k to final stage

(value to go)

Arc value



$f_3(8) = \max\{12{\cdot}1 + f_4(8+3{\cdot}1),\ 12{\cdot}2 + f_4(8+3{\cdot}2)\}$
$= \max\{12, -\infty\} = 12$

$f_4(14) = -\infty$

$f_4(11) = 0$

Recursive optimization

$$\max \ 15x_1 + 10x_2 + 12x_3$$
$$4x_1 + 2x_2 + 3x_3 \leq 12$$
$$x_j \in \{1,2\}$$

The recursion includes arc values:

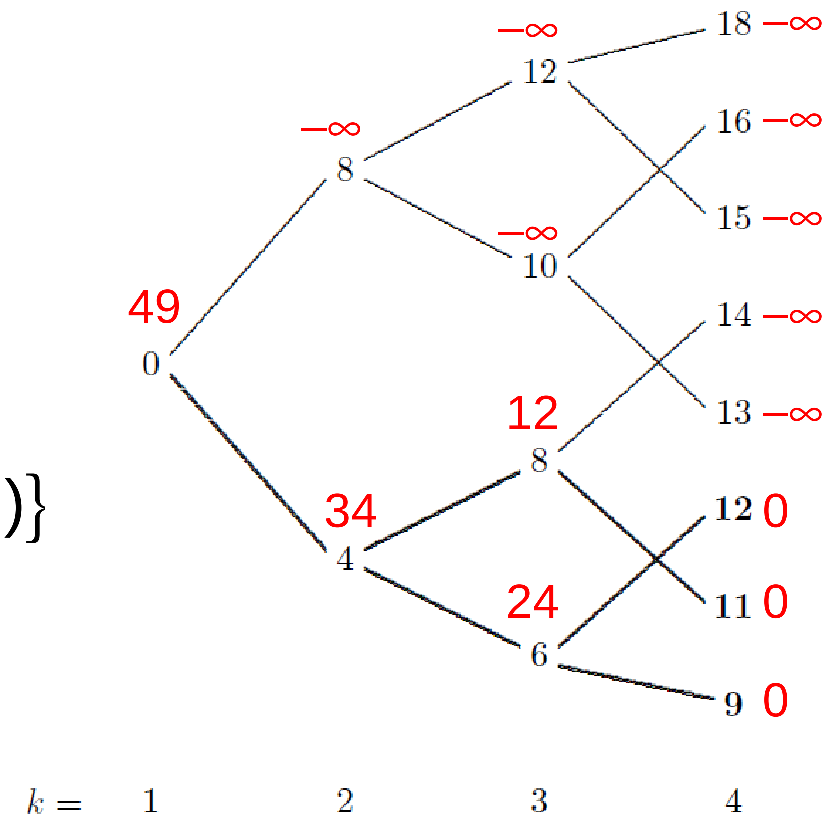
$$f_k(s_k) = \max\{c_k x_k + f_{k+1}(s_k + a_k x_k)\}$$

Boundary condition:

$$f_{n+1}(s_{n+1}) = \begin{cases} 0 & \text{if } s_{n+1} \leq b \\ -\infty & \text{otherwise} \end{cases}$$

Optimal value:

$$f_1(0)$$



$f_k(s_k)$ for each state s_k

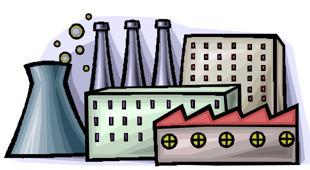
Recursive optimization

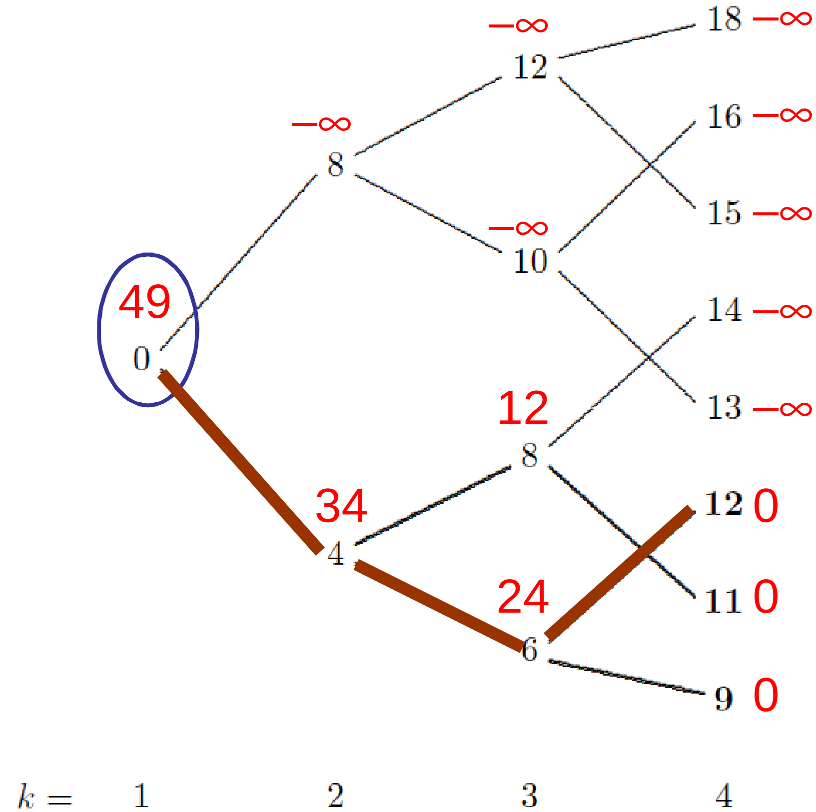
$$\max\ 15x_1 + 10x_2 + 12x_3$$
$$4x_1 + 2x_2 + 3x_3 \le 12$$
$$x_j \in \{1, 2\}$$

The maximum revenue is 49.

The optimal path is easy to retrace.

$(x_1, x_2, x_3) = (1,1,2)$





$f_k(s_k)$ for each state s_k

Filtering: Stretch

Example: $\mathrm{stretch}(x \mid (a, b, c), (2, 2, 2), (3, 3, 3), P)$

where
$x = (x_1, \ldots, x_n)$
pattern $P = \{(a,b), (b,a), (b,c), (c,b)\}$

Shifts must occur in stretches of length 2 or 3.
Workers cannot change directly between shifts *a* and *c*.

Filtering: Stretch

Example: $\mathrm{stretch}(x \mid (a,b,c),(2,2,2),(3,3,3),P)$

where
$x = (x_1, \ldots, x_n)$
pattern $P = \{(a,b), (b,a), (b,c), (c,b)\}$

Shifts must occur in stretches of length 2 or 3.
Workers cannot change directly between shifts *a* and *c*.

Assume variable domains:

x_1	x_2	x_3	x_4	x_5	x_6	x_7
a	a	a		a	a	a
	b	b	b		b	b
c	c		c	c		

Filtering: Stretch

Example: $\text{stretch}(x \mid (a, b, c), (2, 2, 2), (3, 3, 3), P)$

where
$x = (x_1, \ldots, x_n)$
pattern $P = \{(a,b), (b,a), (b,c), (c,b)\}$

Shifts must occur in stretches of length 2 or 3.
Workers cannot change directly between shifts a and c.

Assume variable domains:

x_1	x_2	x_3	x_4	x_5	x_6	x_7
a	a	a		a	a	a
	b	b	b		b	b
c	c		c	c		

One feasible solution.

Filtering: Stretch

Example: $\mathrm{stretch}(x \mid (a,b,c),(2,2,2),(3,3,3),P)$

where
$x = (x_1, \ldots, x_n)$
pattern $P = \{(a,b), (b,a), (b,c), (c,b)\}$

Shifts must occur in stretches of length 2 or 3.
Workers cannot change directly between shifts *a* and *c*.

Assume variable domains:

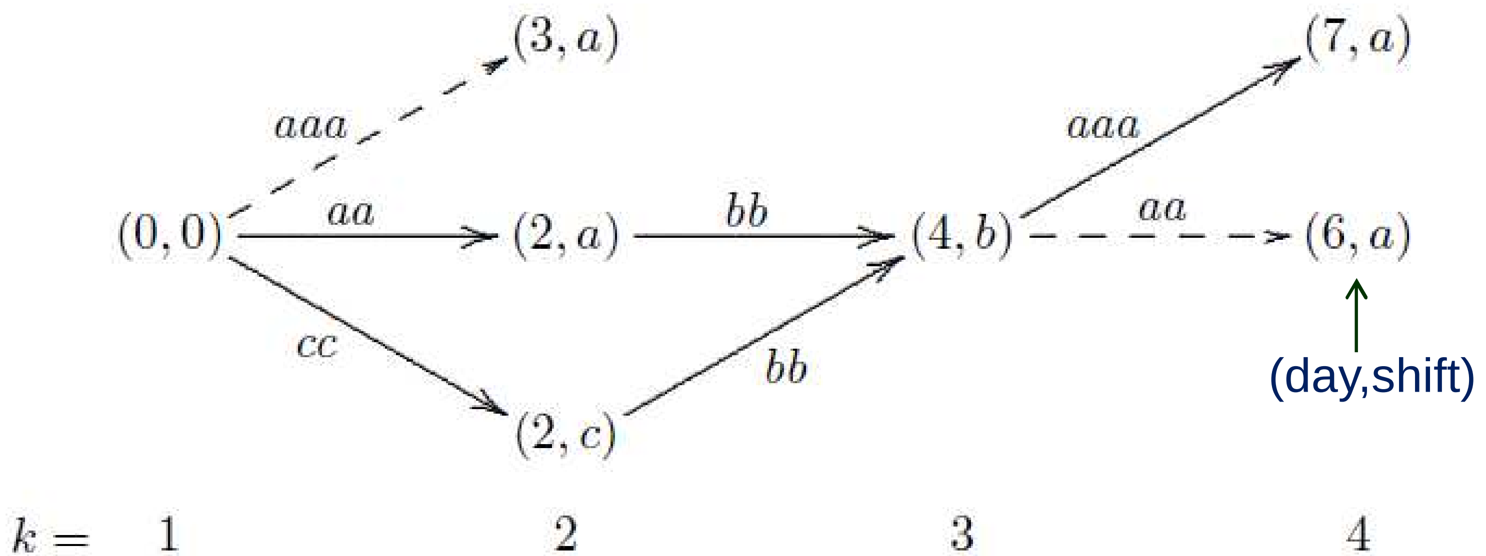
x_1	x_2	x_3	x_4	x_5	x_6	x_7
a	a	a		$\boxed{a}$	$\boxed{a}$	$\boxed{a}$
	b	$\boxed{b}$	$\boxed{b}$		b	b
$\boxed{c}$	$\boxed{c}$		c	c		

The other feasible solution.

Filtering: Stretch

Example: $\mathrm{stretch}(x \mid (a,b,c), (2,2,2), (3,3,3), P)$

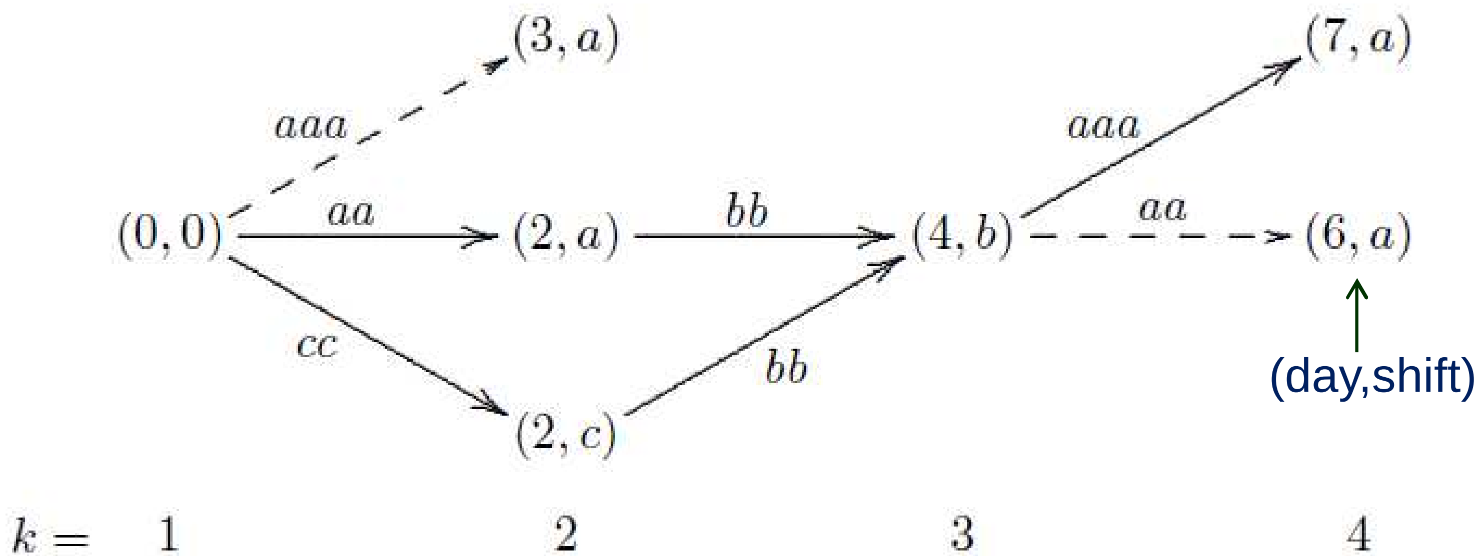
State transition graph (transitions defined by choice of **stretch**):



Filtering: Stretch

Example: $\mathrm{stretch}(x \mid (a,b,c), (2,2,2), (3,3,3), P)$

State transition graph (transitions defined by choice of **stretch**):

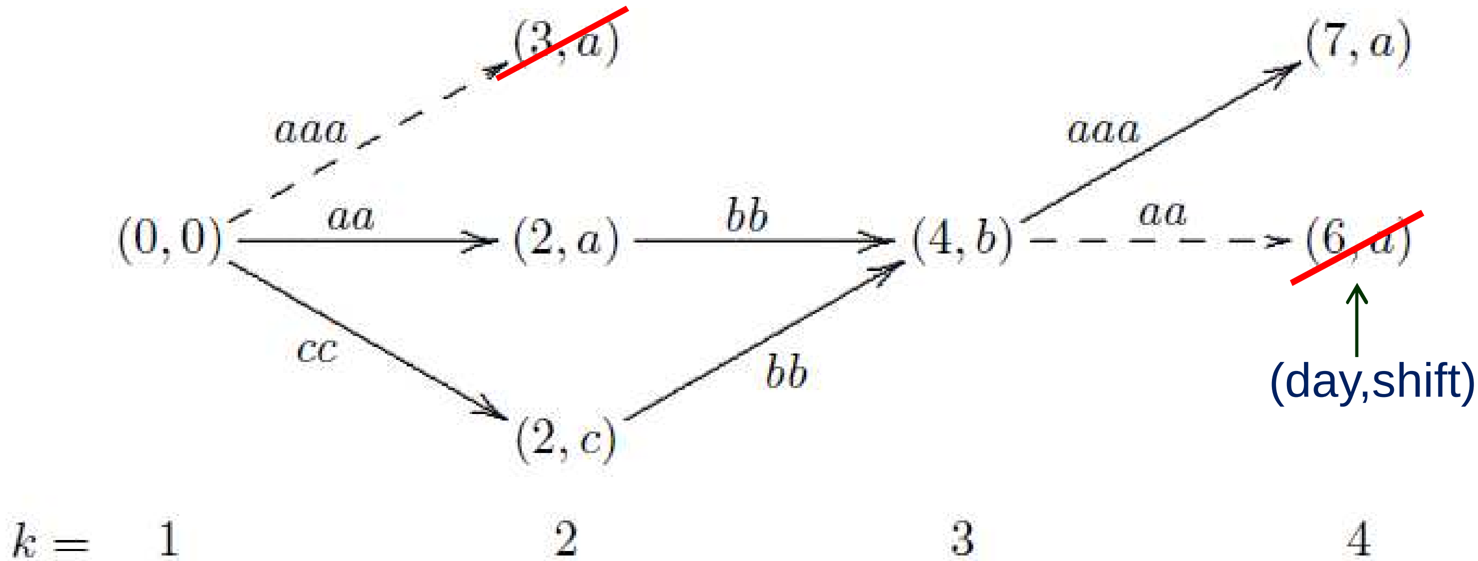


Model is Markovian because pattern constraint involves only 2 consecutive states.

Filtering: Stretch

Example: $\text{stretch}(x \,|\, (a,b,c), (2,2,2), (3,3,3), P)$

State transition graph (transitions defined by choice of **stretch**):

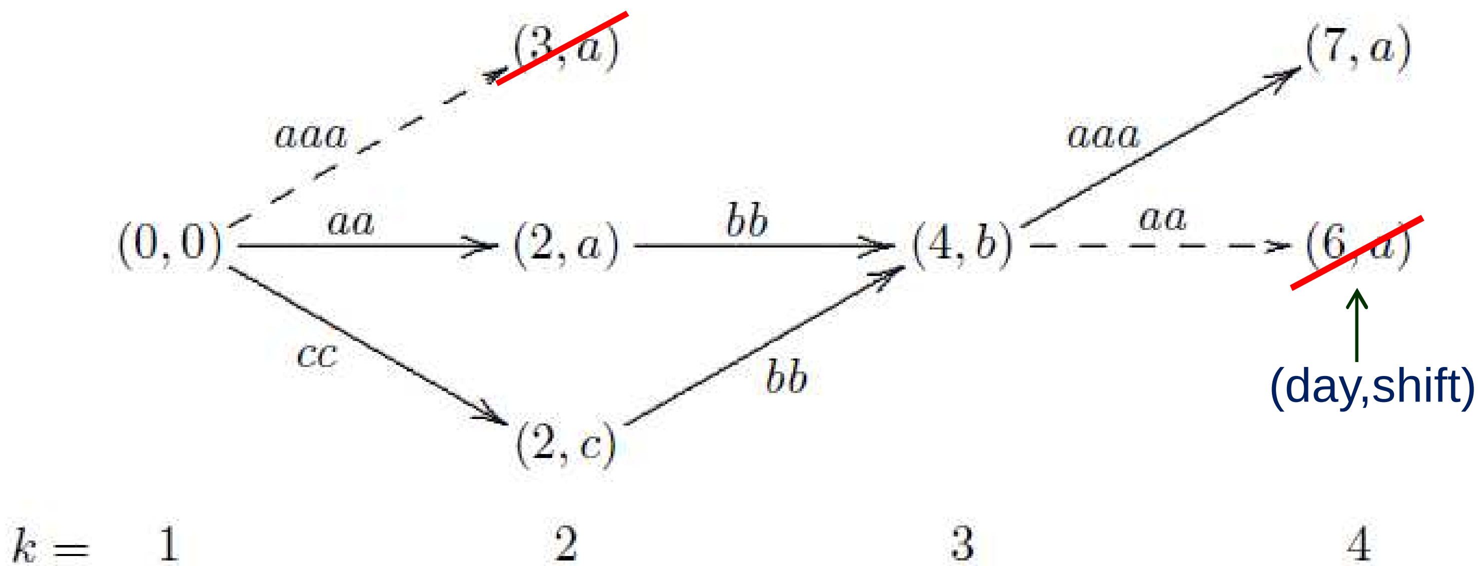


Remove states that are not backward reachable from a feasible end state.

Filtering: Stretch

Example: $\mathrm{stretch}(x \,|\, (a,b,c), (2,2,2), (3,3,3), P)$

State transition graph (transitions defined by choice of **stretch**):



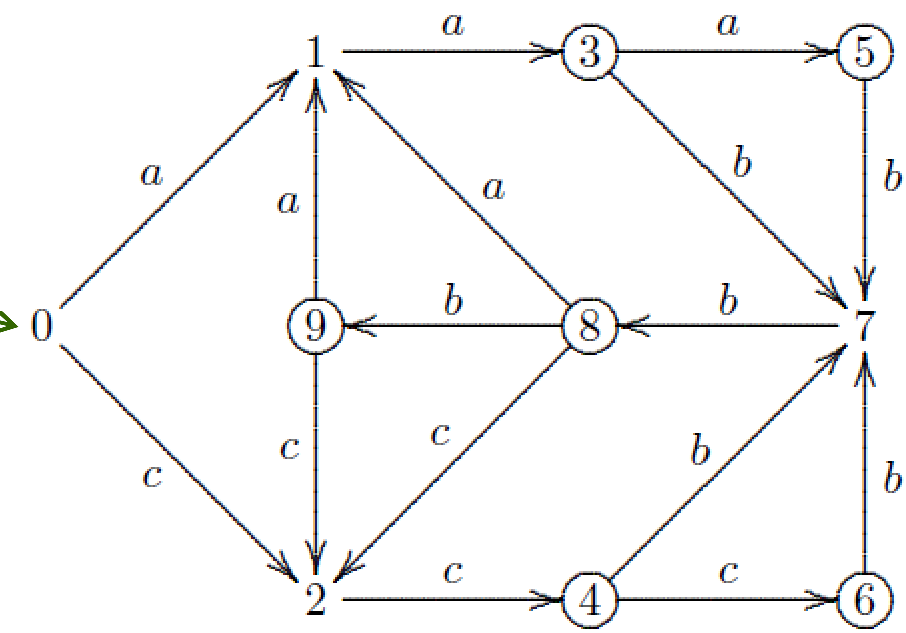
Domains can now be filtered:

x_1	x_2	x_3	x_4	x_5	x_6	x_7
a	a			a	a	a
		b	b			
c	c					

Filtering: Regular

Encode the stretch example as a **finite deterministic automaton *A***:

Initial state



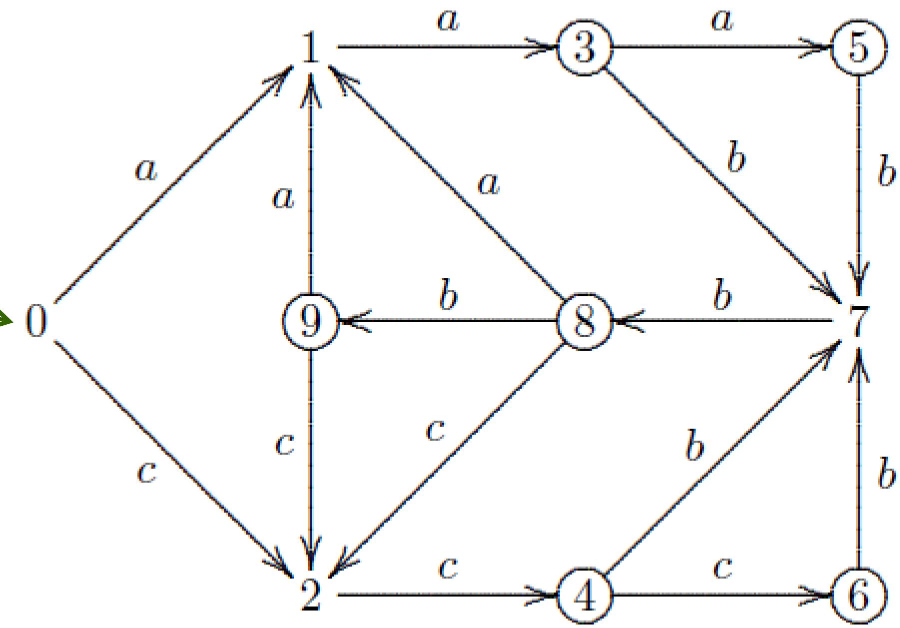
Circled nodes are accepting (terminal) states

Transitions defined by choice of **shift**.

Filtering: Regular

Encode the stretch example as a **finite deterministic automaton *A***:

Initial state



Circled nodes are accepting (terminal) states

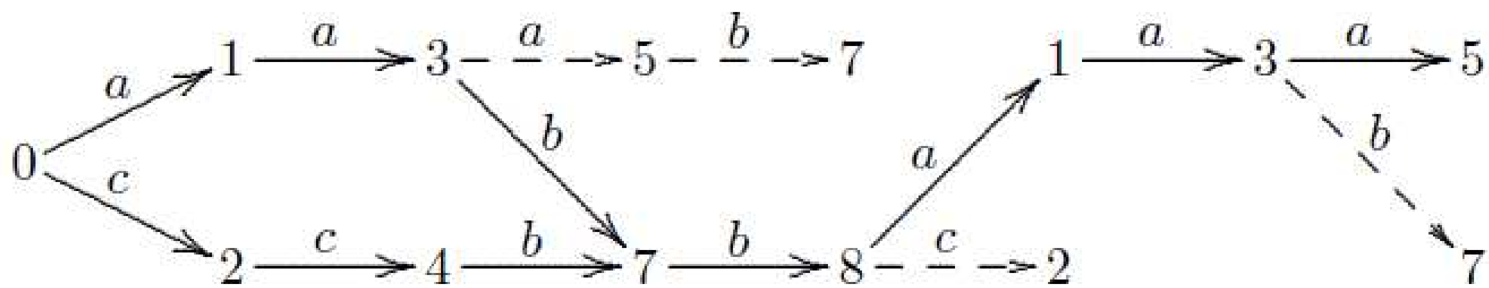
Transitions defined by choice of **shift**.

Now impose the constraint

$$\mathrm{regular}\left((x_1, \ldots, x_7) \mid A\right)$$

Filtering: Regular

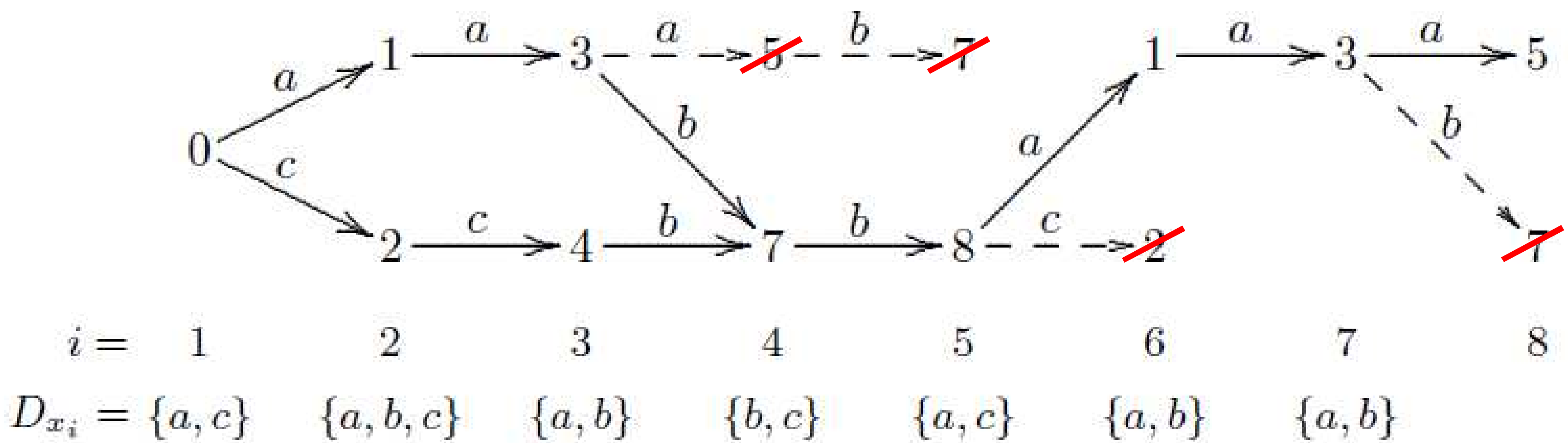
Filtering can be done on a DP state transition graph:



$i =$	1	2	3	4	5	6	7	8
$D_{x_i} =$	$\{a, c\}$	$\{a, b, c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, c\}$	$\{a, b\}$	$\{a, b\}$	

Filtering: Regular

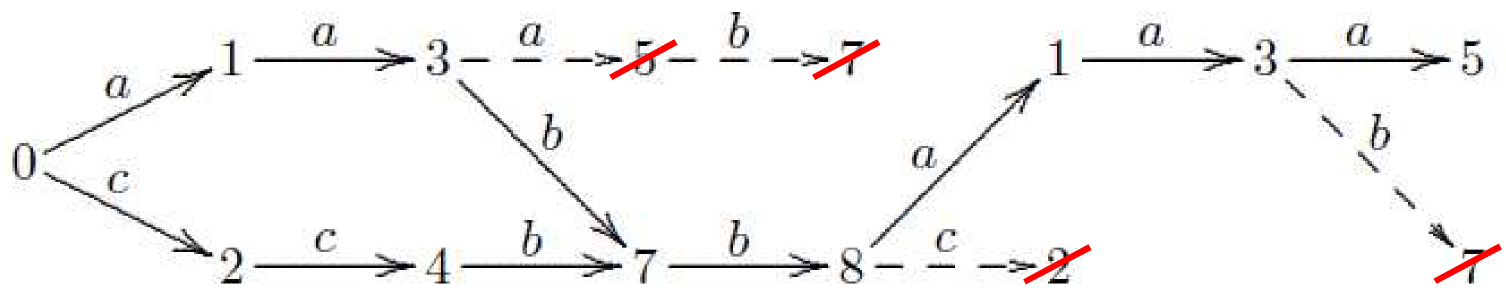
Filtering can be done on a DP state transition graph:



Remove states that are not backward reachable from an accepting state in the final stage.

Filtering: Regular

Filtering can be done on a DP state transition graph:



$i =$	1	2	3	4	5	6	7	8
$D_{x_i} =$	$\{a, c\}$	$\{a, b, c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, c\}$	$\{a, b\}$	$\{a, b\}$	
$D'_{x_i} =$	$\{a, c\}$	$\{a, c\}$	$\{b\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{a\}$	

Remove states that are not backward reachable from an accepting state in the final stage.

Now filter the domains.

CP-based Branch and Price

Basic Idea

Example: Airline Crew Scheduling

Motivation

- **Branch and price** allows solution of integer programming problems with a huge number of variables.
- The problem is solved by a branch-and-relax method. The difference lies in how the LP relaxation is solved.
- Variables are added to the LP relaxation only as needed.
- Variables are **priced** to find which ones should be added.
- **CP** is useful for solving the pricing problem, particularly when constraints are complex.
- **CP-based branch and price** has been successfully applied to airline crew scheduling, transit scheduling, and other transportation-related problems.

Basic Idea

Suppose the LP relaxation of an integer programming problem has a huge number of variables:

$$\begin{aligned} &\min \ cx \\ &Ax = b \\ &x \geq 0 \end{aligned}$$

We will solve a **restricted master problem**, which has a small subset of the variables:

$$\begin{aligned} &\min \ \sum_{j \in J} c_j x_j \\ &\sum_{j \in J} \boxed{A_j} x_j = b \qquad (\lambda) \\ &x_j \geq 0 \end{aligned}$$

Column j of A

Adding x_k to the problem would improve the solution if x_k has a negative reduced cost:

$$r_k = c_k - \lambda A_k < 0$$

Basic Idea

Adding x_k to the problem would improve the solution if x_k has a negative reduced cost:

$$r_k = c_k - \lambda A_k < 0$$

Computing the reduced cost of x_k is known as **pricing** x_k.

So we solve the pricing problem:

Cost of column y

$$\min \ \boxed{c_y} - \lambda y$$
$$y \text{ is a column of } A$$

If the solution y^* satisfies $c_{y^*} - \lambda y^* < 0$, then we can add column y to the restricted master problem.

Basic Idea

The pricing problem $\max \ \lambda y$

y is a column of A

need not be solved to optimality, so long as we find a column with negative reduced cost.

However, when we can no longer find an improving column, we solved the pricing problem to optimality to make sure we have the optimal solution of the LP.

If we can state constraints that the columns of A must satisfy, CP may be a good way to solve the pricing problem.

Example: Airline Crew Scheduling

We want to assign crew members to flights to minimize cost while covering the flights and observing complex work rules.

Flight data

j	s_j	f_j
1	0	3
2	1	3
3	5	8
4	6	9
5	10	12
6	12	14

Start time

Finish time

A **roster** is the sequence of flights assigned to a single crew member.

The gap between two consecutive flights in a roster must be from 2 to 3 hours. Total flight time for a roster must be between 6 and 10 hours.

For example,

flight 1 cannot immediately precede 6

flight 4 cannot immediately precede 5.

The possible rosters are:

(1,3,5), (1,4,6), (2,3,5), (2,4,6)

Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

1	2	3	4
(1,3,5),	(1,4,6),	(2,3,5),	(2,4,6)

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

$$\min\ z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0, \text{ all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

1	2	3	4
(1,3,5),	(1,4,6),	(2,3,5),	(2,4,6)

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

$$\min z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \boxed{1 \quad 1} & & 0 & 0 & \boxed{1 \quad 1} & & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0, \text{ all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

Rosters that cover flight 1.

Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

1	2	3	4
(1,3,5),	(1,4,6),	(2,3,5),	(2,4,6)

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

$$\min z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

$$x_{ik} \geq 0, \text{ all } i, k$$

Rosters that cover flight 2.

Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

1	2	3	4
(1,3,5),	(1,4,6),	(2,3,5),	(2,4,6)

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

$$\min\ z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0, \text{ all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

Rosters that cover flight 3.

Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

1 2 3 4

(1,3,5), (1,4,6), (2,3,5), (2,4,6)

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

$$\min\ z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0, \text{ all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

Rosters that cover flight 4.

Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

1 2 3 4

(1,3,5), (1,4,6), (2,3,5), (2,4,6)

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

$$\min z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0, \text{ all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

Rosters that cover flight 5.

Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

1 2 3 4

(1,3,5), (1,4,6), (2,3,5), (2,4,6)

The LP relaxation of the problem is:

Cost of assigning crew member 1 to roster 2

$$\min\ z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0, \text{ all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

Rosters that cover flight 6.

Airline Crew Scheduling

There are 2 crew members, and the possible rosters are:

1	2	3	4
(1,3,5),	(1,4,6),	(2,3,5),	(2,4,6)

The LP relaxation of the problem is:

Cost c_{12} of assigning crew member 1 to roster 2

$$\min\ z$$

$$\begin{bmatrix} 10 & 12 & 7 & 13 & 9 & 11 & 6 & 12 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_{ik} \geq 0, \text{ all } i, k$$

= 1 if we assign crew member 1 to roster 2, = 0 otherwise.

Each crew member is assigned to exactly 1 roster.

Each flight is assigned at least 1 crew member.

In a real problem, there can be **millions** of rosters.

Airline Crew Scheduling

We start by solving the problem with a subset of the columns:

Optimal dual solution

$$\min\ z$$

$$\begin{bmatrix}10&13&9&12\\1&1&0&0\\0&0&1&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\begin{bmatrix}x_{11}\\x_{14}\\x_{21}\\x_{24}\end{bmatrix}\begin{array}{c}=\\=\\=\\\geq\\\geq\\\geq\\\geq\\\geq\\\geq\end{array}\begin{bmatrix}z\\1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}\qquad\begin{array}{c}\\(10)\\(9)\\(0)\\(0)\\(0)\\(0)\\(0)\\(3)\end{array}\ \begin{array}{l}\\u_1\\u_2\\v_1\\v_2\\v_3\\v_4\\v_5\\v_6\end{array}$$

$$x_{ik}\geq 0,\ \text{all } i,k$$

Airline Crew Scheduling

We start by solving the problem with a subset of the columns:

Dual variables

$$\min\ z$$

$$\begin{bmatrix}10&13&9&12\\1&1&0&0\\0&0&1&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\begin{bmatrix}x_{11}\\x_{14}\\x_{21}\\x_{24}\end{bmatrix}\begin{matrix}=\\=\\=\\\geq\\\geq\\\geq\\\geq\\\geq\\\geq\end{matrix}\begin{bmatrix}z\\1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}\qquad\begin{matrix}\\(10)\\(9)\\(0)\\(0)\\(0)\\(0)\\(0)\\(3)\end{matrix}\quad\begin{matrix}\\u_1\\u_2\\v_1\\v_2\\v_3\\v_4\\v_5\\v_6\end{matrix}$$

$$x_{ik}\geq 0,\ \text{all } i,k$$

Airline Crew Scheduling

We start by solving the problem with a subset of the columns:

$$\min\ z$$

$$\begin{bmatrix} 10 & 13 & 9 & 12 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{14} \\ x_{21} \\ x_{24} \end{bmatrix} \begin{array}{c} = \\ = \\ = \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \\ \geq \end{array} \begin{bmatrix} z \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{array}{cc} & \text{Dual variables} \\ (10) & u_1 \\ (9) & u_2 \\ (0) & v_1 \\ (0) & v_2 \\ (0) & v_3 \\ (0) & v_4 \\ (0) & v_5 \\ (3) & v_6 \end{array}$$

$$x_{ik} \geq 0,\ \text{all } i, k$$

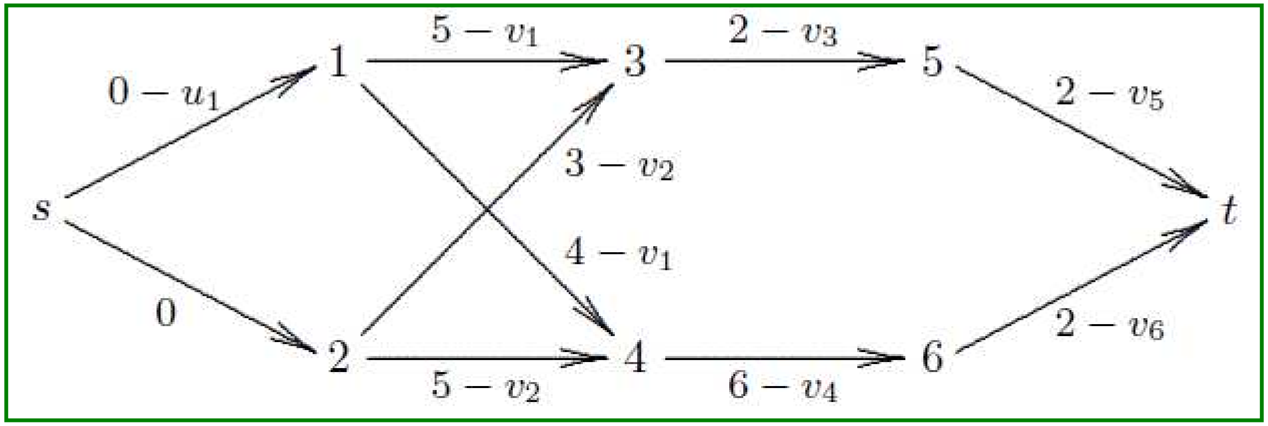
The reduced cost of an excluded roster *k* for crew member *i* is

$$c_{ik} - u_i - \sum_{j \text{ in roster k}} v_j$$

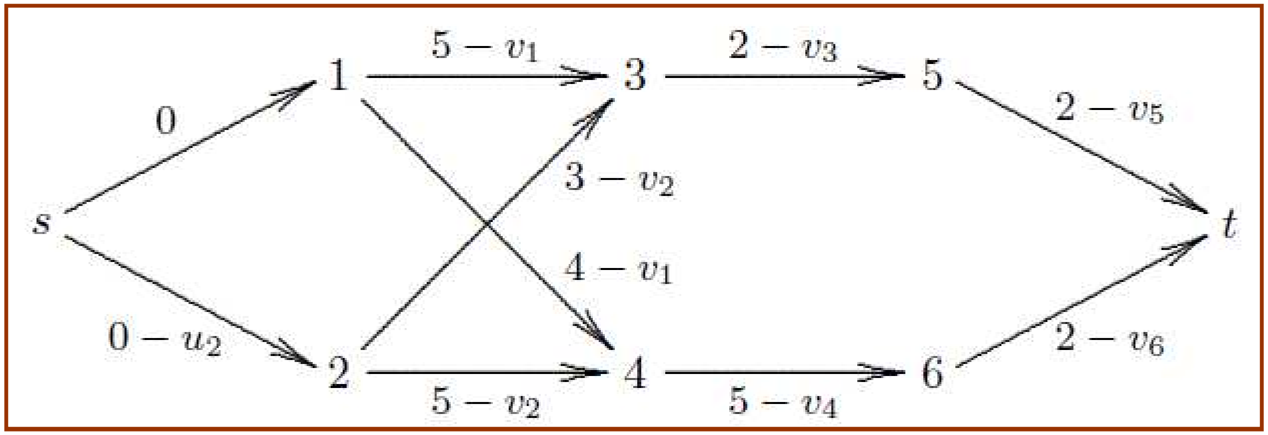
We will formulate the pricing problem as a shortest path problem.

Pricing problem

Crew member 1



Crew member 2

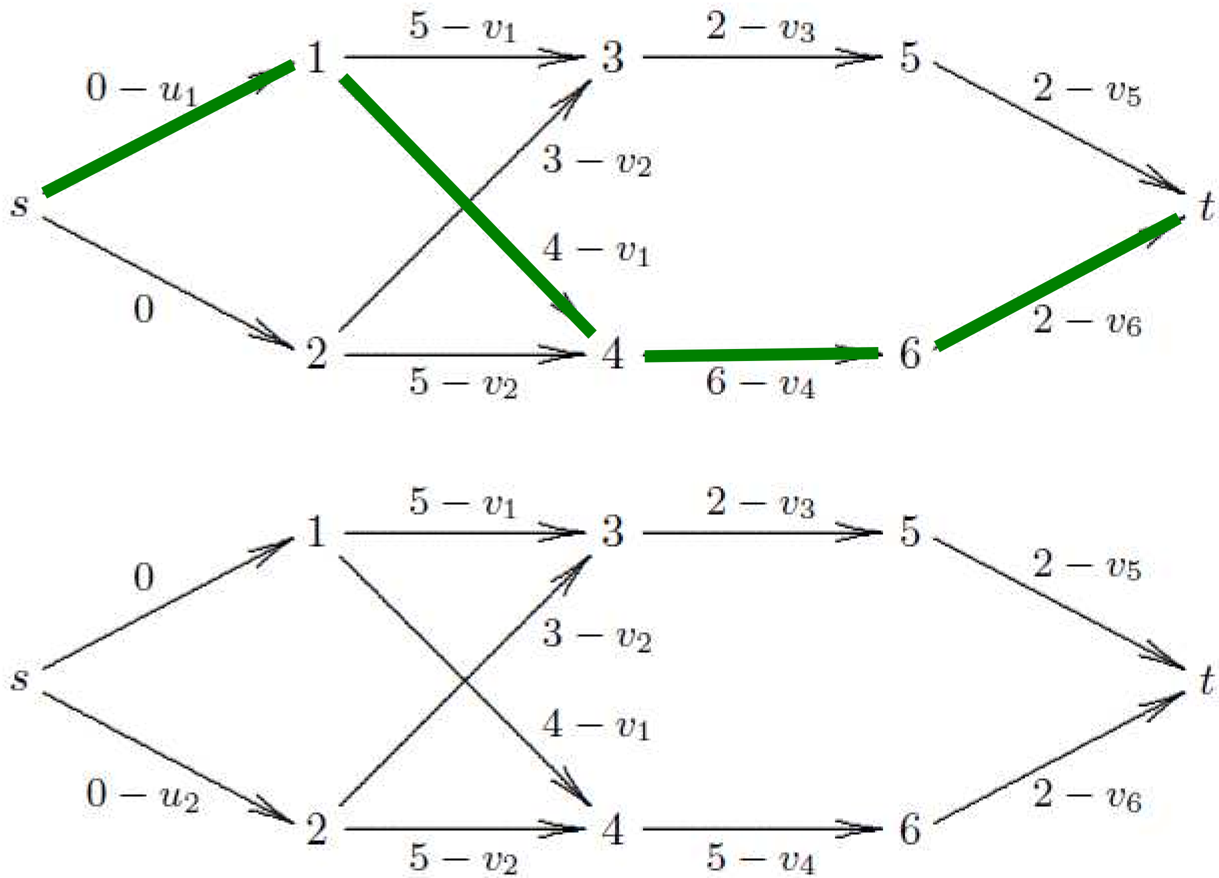


Pricing problem

Each s-t path corresponds to a roster, provided the flight time is within bounds.

Crew member 1

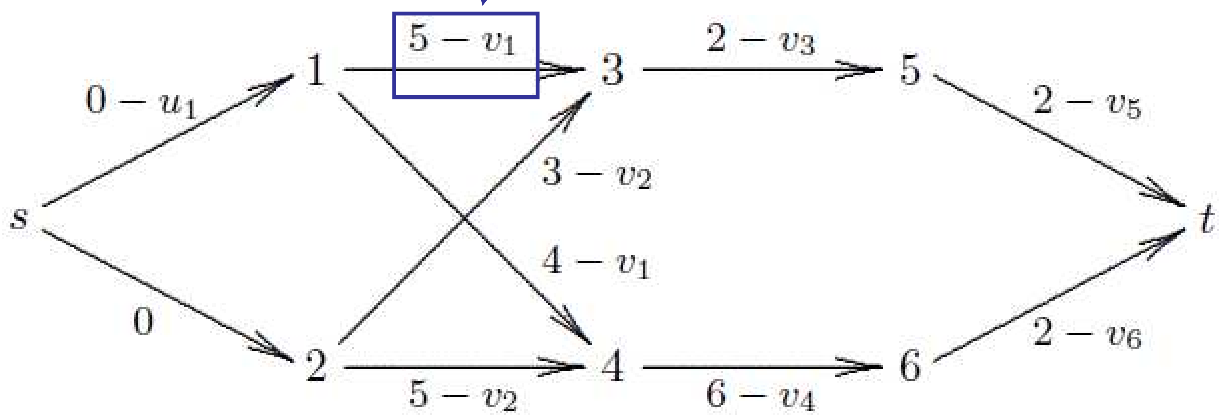
Crew member 2



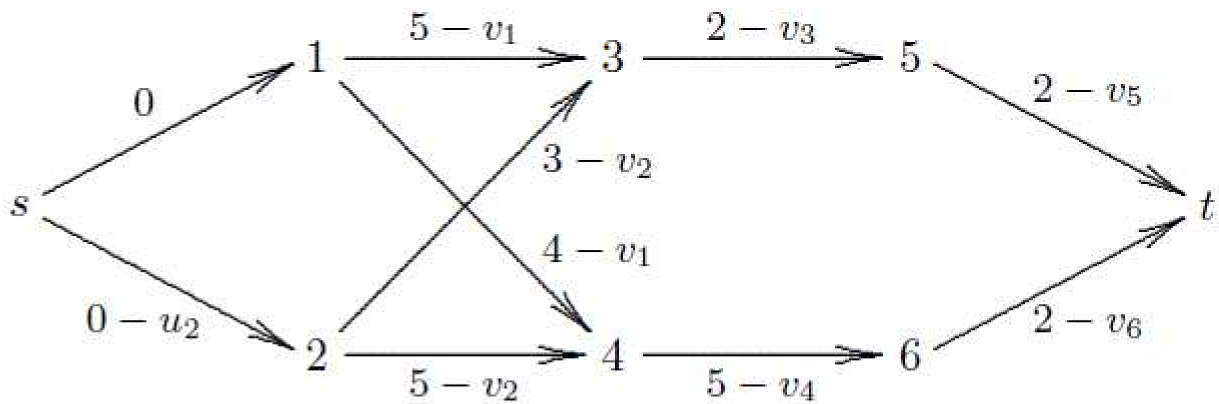
Pricing problem

Cost of flight 3 if it immediately follows flight 1, offset by dual multiplier for flight 1

Crew member 1



Crew member 2



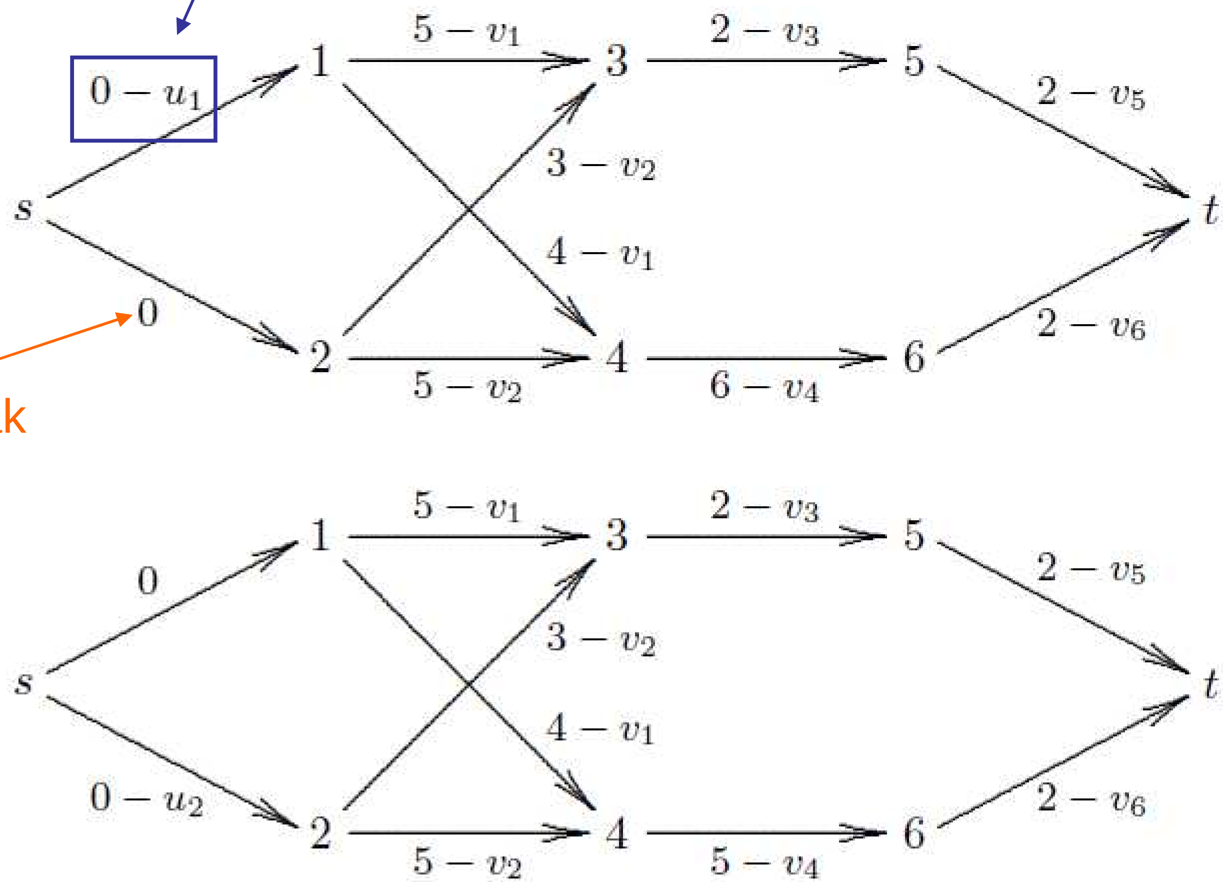
Pricing problem

Cost of transferring from home to flight 1, offset by dual multiplier for crew member 1

Crew member 1

Dual multiplier omitted to break symmetry

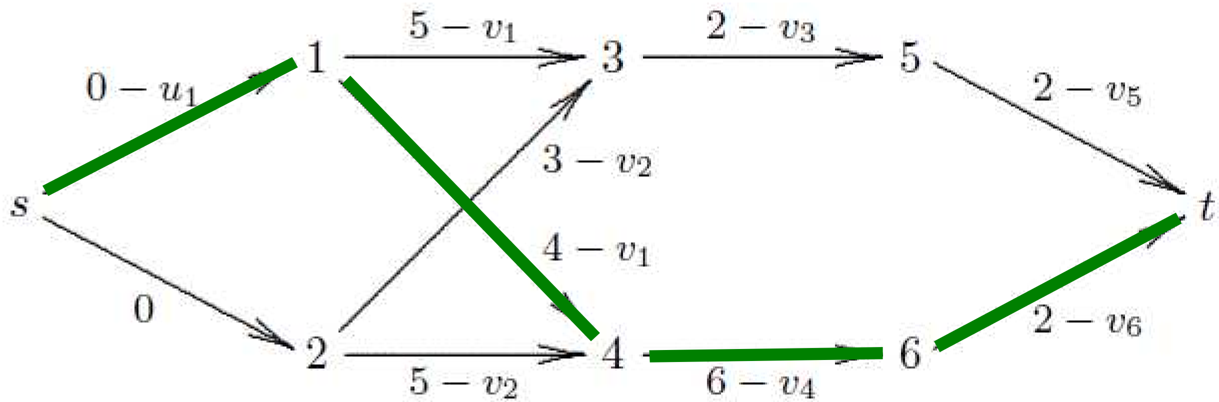
Crew member 2



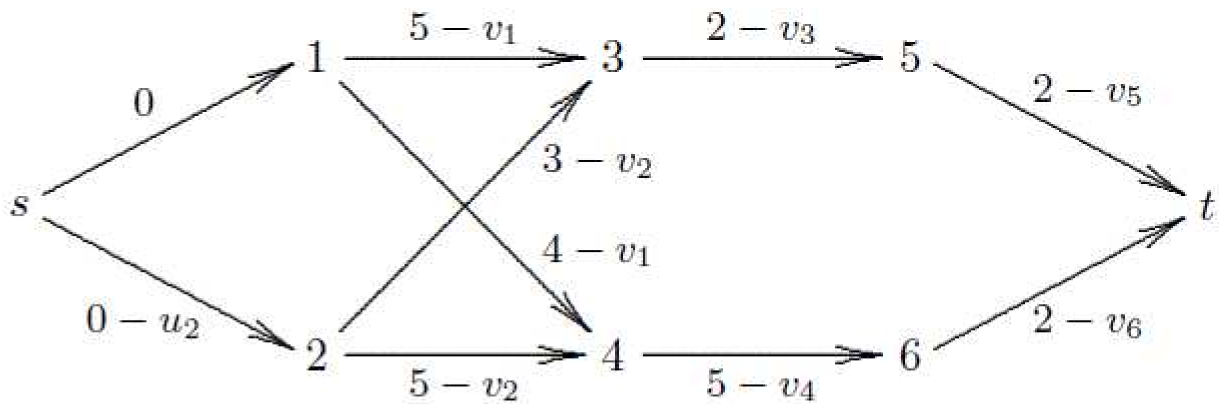
Pricing problem

Length of a path is reduced cost of the corresponding roster.

Crew member 1



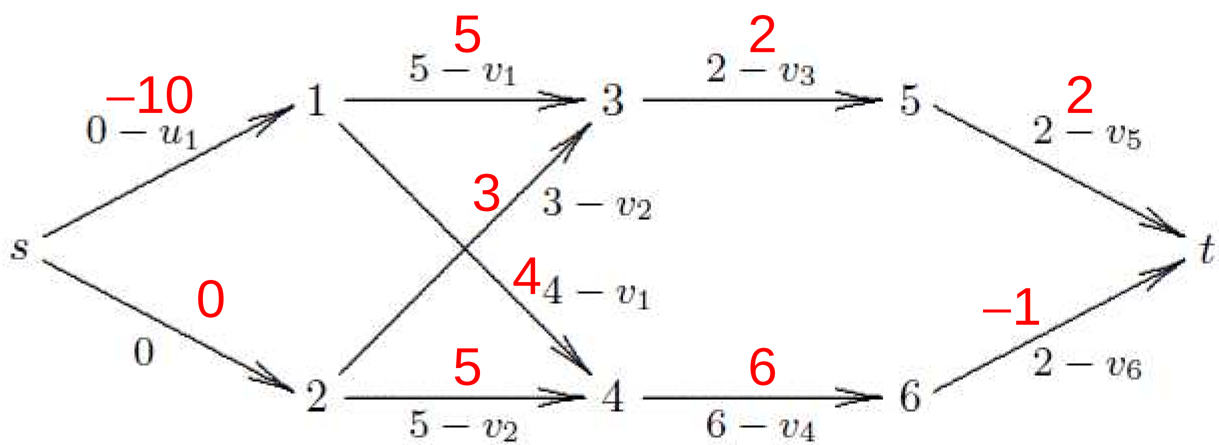
Crew member 2



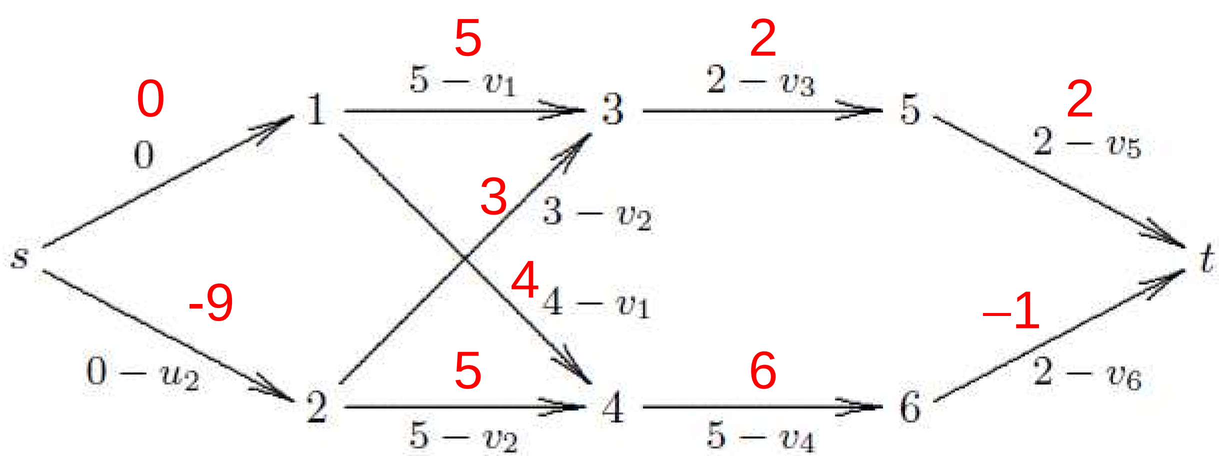
Pricing problem

Arc lengths using dual solution of LP relaxation

Crew member 1



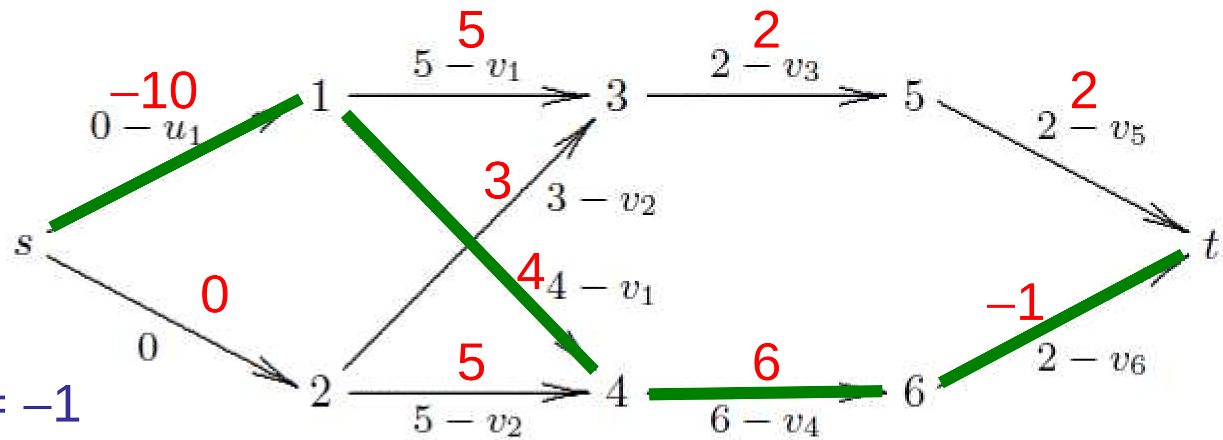
Crew member 2



Pricing problem

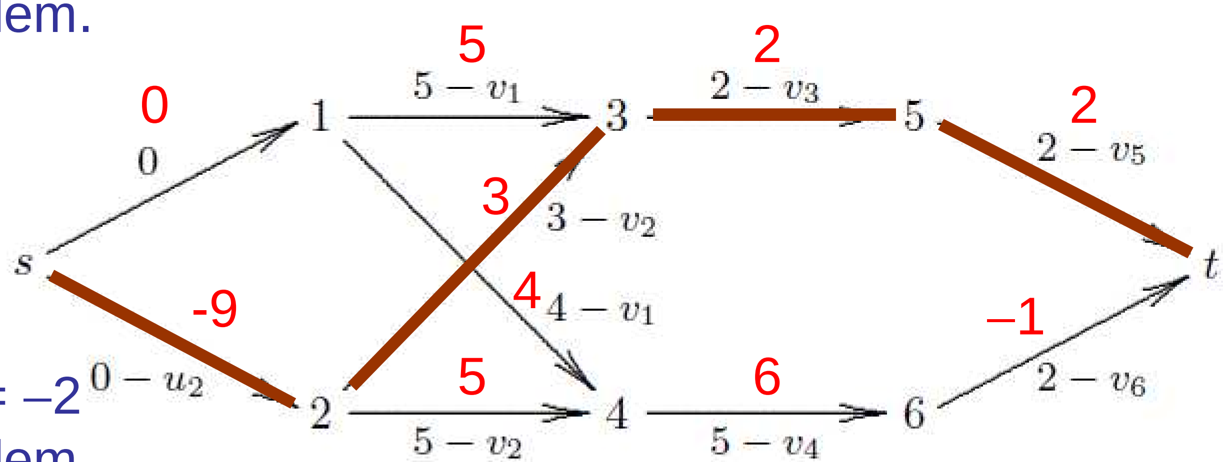
Solution of shortest path problems

Crew
member 1



Reduced cost = –1
Add x_{12} to problem.

Crew
member 2



Reduced cost = –2
Add x_{23} to problem.

After x_{12} and x_{23} are added to the problem, no remaining variable has negative reduced cost.

Pricing problem

The shortest path problem cannot be solved by traditional shortest path algorithms, due to the bounds on total duration of flights.

It **can** be solved by CP:

Set of flights assigned to crew member *i*

Path length

Graph

Path global constraint

$$\text{Path}(\boxed{X_i}, \boxed{z_i}, \boxed{G}), \text{ all flights } i$$

Setsum global constraint

$$T_{\min} \le \sum_{j \in X_i} \left(\boxed{f_j - s_j}\right) \le T_{\max}$$

$$X_i \subset \{\text{flights}\}, \quad z_i < 0, \text{ all } i$$

Duration of flight *j*

CP-based Benders Decomposition

Benders Decomposition in the Abstract

Classical Benders Decomposition

Example: Machine Scheduling

Motivation

- **Benders decomposition** allows us to apply CP and OR to different parts of the problem.

- It searches over values of certain variables that, when fixed, result in a much simpler **subproblem**.

- The search learns from past experience by accumulating **Benders cuts** (a form of nogood).

- The technique can be **generalized** far beyond the original OR conception.

- Generalized Benders methods have resulted in the **greatest speedups** achieved by combining CP and OR.

- Instance of constraint-directed search.

- Generates constraints (nogoods) by solving **inference dual** of subproblem.

Benders Decomposition in the Abstract

Benders decomposition can be applied to problems of the form

$$\min\ f(x,y)$$
$$S(x,y)$$
$$x \in D_x,\ \ y \in D_y$$

When x is fixed to some value, the resulting **subproblem** is much easier:

$$\min\ f(\bar{x},y)$$
$$S(\bar{x},y)$$
$$y \in D_y$$

…perhaps because it decouples into smaller problems.

For example, suppose *x* assigns jobs to machines, and *y* schedules the jobs on the machines.

When *x* is fixed, the problem decouples into a separate scheduling subproblem for each machine.

Benders Decomposition

We will search over assignments to x. This is the **master problem**.

In iteration k we assume $x = x^k$ and solve the subproblem

$$\begin{aligned} &\min\ f(x^k, y) \\ &S(x^k, y) \\ &y \in D_y \end{aligned}$$

and get optimal value v_k

We generate a **Benders cut** (a type of nogood) $\boxed{v} \geq B_{k+1}(x)$

that satisfies $B_{k+1}(x^k) = v_k$.

Cost in the original problem

The Benders cut says that if we set $x = x^k$ again, the resulting cost v will be at least v_k. To do better than v_k, we must try something else.

It also says that any other x will result in a cost of at least $B_{k+1}(x)$, perhaps due to some similarity between x and x^k.

Benders Decomposition

We will search over assignments to *x*. This is the **master problem**.

In iteration *k* we assume $x = x^k$ and solve the subproblem

$$\begin{aligned} &\min\ f(x^k, y) \\ &S(x^k, y) \\ &y \in D_y \end{aligned}$$

and get optimal value v_k

We generate a **Benders cut** (a type of nogood) $\boxed{v} \geq B_{k+1}(x)$

that satisfies $B_{k+1}(x) = v_k$.

Cost in the original problem

We add the Benders cut to the master problem, which becomes

$$\begin{aligned} &\min\ v \\ &v \geq B_i(x),\ i = 1, \ldots, k+1 \\ &x \in D_x \end{aligned}$$

Benders cuts generated so far

Benders Decomposition

We now solve the master problem

$$\begin{aligned} &\min\ v \\ &v \geq B_i(x),\ i = 1,\ldots,k+1 \\ &x \in D_x \end{aligned}$$

to get the next trial value x^{k+1}.

The master problem is a relaxation of the original problem, and its optimal value is a **lower bound** on the optimal value of the original problem.

The subproblem is a restriction, and its optimal value is an **upper bound**.

The process continues until the bounds meet.

The Benders cuts partially define the **projection** of the feasible set onto *x*. We hope not too many cuts are needed to find the optimum.

Classical Benders Decomposition

The classical method applies to problems of the form

$$\begin{aligned} &\min\ f(x) + cy \\ &g(x) + Ay \ge b \\ &x \in D_x,\ y \ge 0 \end{aligned}$$

and the subproblem is an LP

$$\begin{aligned} &\min\ f(x^k) + cy \\ &Ay \ge b - g(x^k) \quad (\lambda) \\ &y \ge 0 \end{aligned}$$

whose dual is

$$\begin{aligned} &\max\ f(x^k) + \lambda\left(b - g(x^k)\right) \\ &\lambda A \le c \\ &\lambda \ge 0 \end{aligned}$$

Let λ^k solve the dual.

By strong duality, $B_{k+1}(x) = f(x) + \lambda^k(b - g(x))$ is the tightest lower bound on the optimal value v of the original problem when $x = x^k$.

Even for other values of x, λ^k **remains feasible in the dual**. So by weak duality, $B_{k+1}(x)$ remains a lower bound on v.

Classical Benders

So the master problem becomes

$$
\begin{aligned}
&\min\ v \\
&v \geq B_i(x),\ i = 1,\ldots,k+1 \\
&x \in D_x
\end{aligned}
\qquad\qquad
\begin{aligned}
&\min\ v \\
&v \geq f(x) + \lambda^i(b - g(x)),\ i = 1,\ldots,k+1 \\
&x \in D_x
\end{aligned}
$$

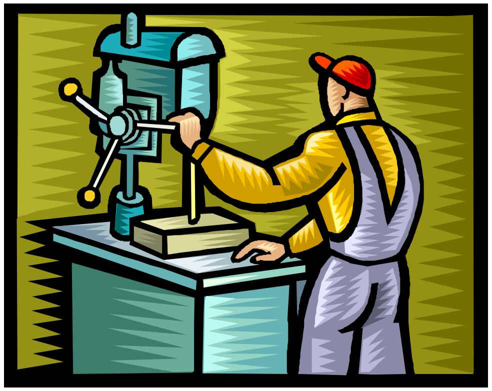
In most applications the master problem is

- an MILP
- a nonlinear programming problem (NLP), or
- a mixed integer/nonlinear programming problem (MINLP).

Example: Machine Scheduling

- Assign 5 jobs to 2 machines (A and B), and schedule the machines assigned to each machine within time windows.
- The objective is to minimize **makespan**.

Time lapse between start of first job and end of last job.



- Assign the jobs in the **master problem**, to be solved by **MILP**.
- Schedule the jobs in the **subproblem**, to be solved by **CP**.

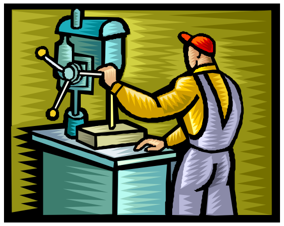
Machine Scheduling

Job Data

Job j	Release time r_j	Dead-line d_j	Processing time p_{Aj}	p_{Bj}
1	0	10	1	5
2	0	10	3	6
3	2	7	3	7
4	2	10	4	6
5	4	7	2	5

Machine A

Machine B



Once jobs are assigned, we can minimize overall makespan by minimizing makespan on each machine individually.

So the subproblem decouples.

Machine Scheduling

Job Data

Job j	Release time r_j	Dead-line d_j	Processing time p_{Aj}	p_{Bj}
1	0	10	1	5
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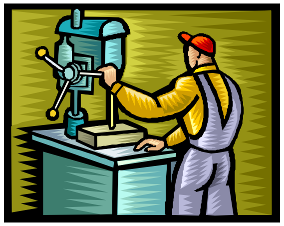
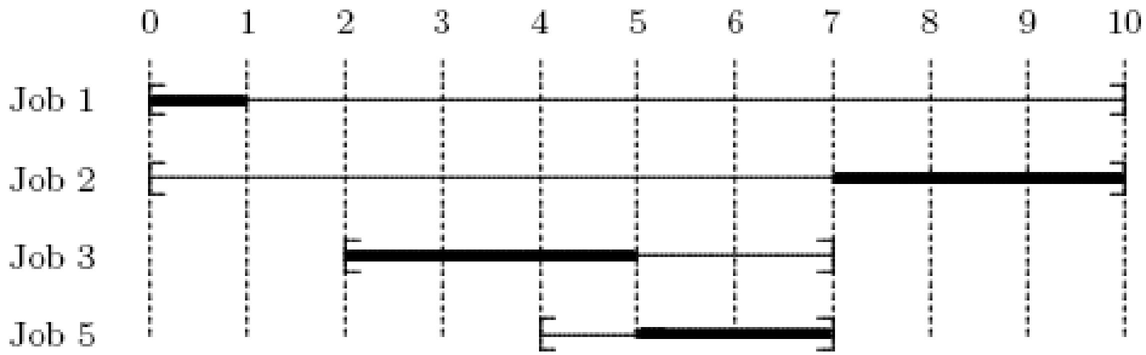
3	2	7	3	7
4	2	10	4	6
5	4	7	2	5

Once jobs are assigned, we can minimize overall makespan by minimizing makespan on each machine individually.

So the subproblem decouples.



Minimum makespan schedule for jobs 1, 2, 3, 5 on machine A



Machine Scheduling

The problem is

Start time of job *j*

$$\min \ M$$

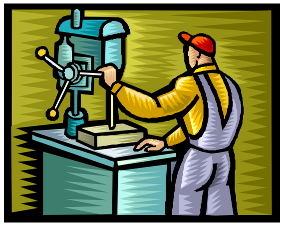
$$M \geq \boxed{s_j} + p_{x_j j}, \text{ all } j$$

Time windows

$$r_j \leq s_j \leq d_j - p_{x_j j}, \text{ all } j$$

Jobs cannot overlap

$$\text{noOverlap}\big((s_j \mid x_j = i), (p_{ij} \mid x_j = i)\big), \text{ all } i$$



Machine Scheduling

The problem is

Start time of job *j*

$$\min \; M$$

$$M \geq \boxed{s_j} + p_{x_j j}, \text{ all } j$$

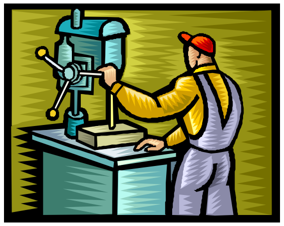
Time windows

$$r_j \leq s_j \leq d_j - p_{x_j j}, \text{ all } j$$

Jobs cannot overlap

$$\text{noOverlap}\big((s_j \,|\, x_j = i), (p_{ij} \,|\, x_j = i)\big), \text{ all } i$$

For a fixed assignment $\bar{x}$ the subproblem on each machine *i* is



$$\min \; M$$

$$M \geq s_j + p_{\bar{x}_j j}, \text{ all } j \text{ with } \bar{x}_j = i$$

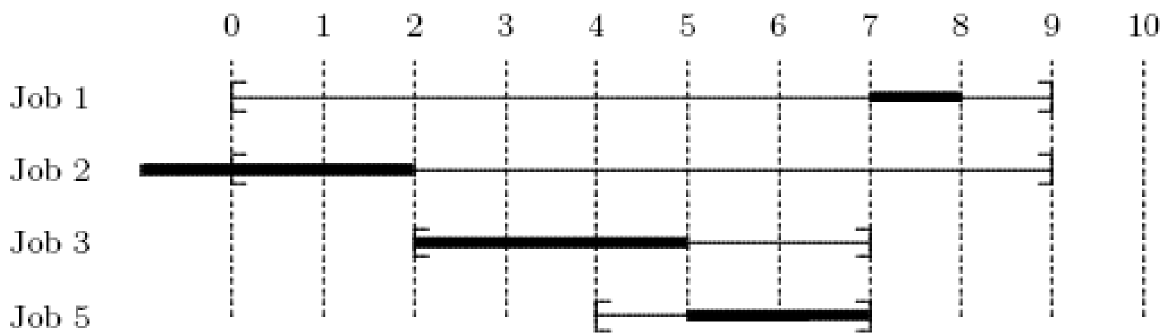
$$r_j \leq s_j \leq d_j - p_{\bar{x}_j j}, \text{ all } j \text{ with } \bar{x}_j = i$$

$$\text{noOverlap}\big((s_j \,|\, \bar{x}_j = i), (p_{ij} \,|\, \bar{x}_j = i)\big)$$

Benders cuts

Suppose we assign jobs 1,2,3,5 to machine A in iteration *k*.

We can prove that 10 is the optimal makespan by proving that the schedule is infeasible with makespan 9.



Edge finding derives infeasibility by reasoning only with jobs 2,3,5. So these jobs alone create a minimum makespan of 10.

So we have a Benders cut

$$v \ge B_{k+1}(x) = \begin{cases} 10 & \text{if } x_2 = x_3 = x_4 = A \\ 0 & \text{otherwise} \end{cases}$$

Benders cuts

We want the master problem to be an MILP, which is good for assignment problems.

So we write the Benders cut

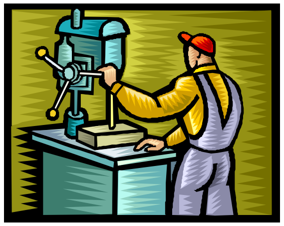
$$v \ge B_{k+1}(x) = \begin{cases} 10 & \text{if } x_2 = x_3 = x_4 = A \\ 0 & \text{otherwise} \end{cases}$$

Using 0-1 variables:

$$v \ge 10\left(x_{A2} + x_{A3} + \boxed{x_{A5}} - 2\right)$$

$$v \ge 0$$

= 1 if job 5 is assigned to machine A



Master problem

The master problem is an MILP:

$$\min \; v$$

$$\sum_{j=1}^{5} p_{Aj} x_{Aj} \le 10, \text{ etc.}$$

$$\sum_{j=1}^{5} p_{Bj} x_{Bj} \le 10, \text{ etc.}$$

Subproblem relaxation derived from time windows

$$v \ge \sum_{j=1}^{5} p_{ij} x_{ij}, \quad v \ge 2 + \sum_{j=3}^{5} p_{ij} x_{ij}, \text{ etc.}, \quad i = A,B$$

Subproblem relaxation derived from release times

$$\mathrm{v} \ge 10(x_{A2} + x_{A3} + x_{A5} - 2)$$

Benders cut from machine A

$$v \ge 8x_{B4}$$

Benders cut from machine B

$$x_{ij} \in \{0,1\}$$

Cumulative scheduling subproblem

Benders cut for **min makespan (**all release times the same):

$$v \geq M_{ik}\left(\sum_{j\in J_{ik}} p_{ij}(1-x_{ij}) + \max_{j\in J_{ik}}\{d_j\} - \min_{j\in J_{ik}}\{d_j\}\right)$$

Min makespan
on machine *i*
in iteration *k*

Set of jobs
assigned to
machine *i* in
iteration *k*

Cumulative scheduling subproblem

Benders cut for **min total tardiness** :

$$v \geq T_{ik}\left(1 - \sum_{j \in J_{ik}} (1 - x_{ij})\right), \quad \text{all } i$$

$$v \geq T_{ik}^{0}\left(1 - \sum_{j \in J_{ik} \setminus Z_{ik}} (1 - x_{ij})\right), \quad \text{all } i$$

Min total tardiness on machine *i* in iteration *k*

Min tardiness when all jobs in Z_{ik} are removed from machine *i*

Set of jobs that, when individually removed from J_{ik}, do not reduce min tardiness

Cumulative scheduling subproblem

Because the tardiness Benders cuts are weak, a good **subproblem relaxation** is particularly important:

$$v \geq \sum_{\ell} T_{\ell k}$$

$$T_{\ell k} = \left(\frac{1}{C_i} \sum_{j=1}^{\ell} p_{i\pi_i(j)} c_{i\pi_i(j)} - d_\ell \right)^+, \quad \ell = 1, \ldots, n$$

Capacity of machine *i*

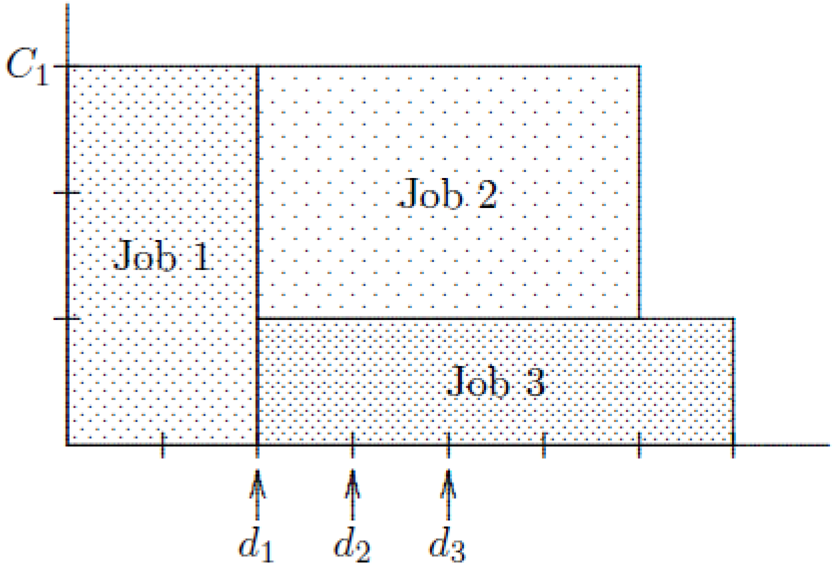
Rate of resource consumption of job *j*

where

$$p_{i\pi_i(1)} c_{i\pi_i(1)} \leq \cdots \leq p_{i\pi_i(n)} c_{i\pi_i(n)}$$

Cumulative scheduling subproblem

Example



$$v \ge \sum_{\ell} T_{\ell k}$$

$$T_{\ell k} = \left(\frac{1}{C_i} \sum_{j=1}^{\ell} p_{i\pi_i(j)} c_{i\pi_i(j)} - d_\ell \right)^+, \quad \ell = 1, \ldots, n$$

$$(\pi_1(1), \pi_1(2), \pi_1(3)) = (3,1,2)$$

Relaxation:

$$v \ge T_{1k} + T_{2k} + T_{3k}$$

$$T_{1k} = \left(\tfrac{1}{3}(5) - 2\right)^+ = 0$$

$$T_{2k} = \left(\tfrac{1}{3}(5+6) - 3\right)^+ = \tfrac{2}{3}$$

$$T_{3k} = \left(\tfrac{1}{3}(5+6+8) - 4\right)^+ = 2\tfrac{1}{3}$$

Bound = 3

Min tardiness = 6

Some Topics Not Covered

- Polyhedral relaxations for metaconstraints (alldiff, element, circuit, noOverlap, cumulative, logic, etc.)
- MILP models for metaconstraints.
- Unifying role of inference duality in constraint-based search (e.g., Benders, DPLL, tabu search).
- Unification of exhaustive and local search.
- Constraint store as relaxation (e.g., relaxed multivalued decision diagram).


THE
END